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On quasi-Einstein sequential warped product manifolds

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ABSTRACT

We find the necessary and sufficient conditions for a sequential warped product manifold to be a quasi-Einstein manifold. We also investigate the necessary conditions for a sequential standard static space-time and a sequential generalized Robertson-Walker space-time to be a manifold of quasi-constant curvature.

1. Introduction

A (pseudo)-Riemannian manifold (M, g), $n \geq 2$, is said to be an *Einstein manifold* [18], if its Ricci tensor Ric is of the form $Ric = \alpha g$, where $\alpha \in C^{\infty}(M)$. It is well-known that if $n > 2$, then α is a constant. Let (M, g), $n > 2$, be a Riemannian manifold. Then the manifold (M, g) is defined to be a *quasi-Einstein manifold* [5], if the condition

$$Ric(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) \tag{1.1}$$

is satisfied on M, where α and β are some differentiable functions on M with $\beta \neq 0$ and A is a non-zero 1-form defined as

$$g(X, U) = A(X), \tag{1.2}$$

for any vector field $X \in \chi(M)$ and a unit vector field U on M. If $\beta = 0$, then the manifold turns into an Einstein manifold. The notion of a quasi-Einstein manifold arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of conformally flat spaces [11], [16].

Let (M, g), $n > 2$, be a (pseudo)-Riemannian manifold. Then the manifold (M, g) is defined to be *a manifold of quasi-constant curvature* [6], if its curvature tensor field R satisfies

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$$R(X,Y,Z,W)=a\left[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\right]$$
$$+b\left[g(X,W)A(Y)A(Z)-g(X,Z)A(Y)A(W)+g(Y,Z)A(X)A(W)-g(Y,W)A(X)A(Z)\right], \tag{1.3}$$

where a, b are real valued functions on M such that $b \neq 0$ and A is a 1-form defined by (1.2). If $b = 0$, then M is a space of constant curvature. It is easy to see that every (pseudo)-Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

The notion of a warped product was introduced by Bishop and O'Neill in [4] to construct Riemannian manifolds with negative sectional curvature. Warped products have remarkable applications in differential geometry as well as in mathematical physics, especially in general relativity.

Warped products have some generalizations like doubly warped products [24] and multiply warped products [23]. A new generalization of warped products is named as *sequential warped product* [12], where the base factor of the warped product is itself a new warped product manifold. Let (M_i, g_i) be three pseudo-Riemannian manifolds with metrics g_i and dimensions m_i for $i = 1, 2, 3$. Let $f : M_1 \to (0,\infty)$ and $h : M_1 \times M_2 \to (0,\infty)$ be two smooth positive functions on M_1 and $M_1 \times M_2$, respectively. Then, the sequential warped product manifold $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ is the triple product manifold $\overline{M} = (M_1 \times M_2) \times M_3$ furnished with the metric tensor

$$\overline{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3, \tag{1.4}$$

(see [12]). Taub-Nut and stationary metrics, Schwarzschild and generalized Riemannian anti de Sitter $\mathbb{T}^2$ black hole metrics are non-trivial examples of sequential warped products [12].

Notation 1.1. *[12] Let $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ be a sequential warped product with metric $\overline{g} = g_1 \oplus f^2 g_2 \oplus h^2 g_3$, where $f : M_1 \to (0,\infty)$ and $h : M_1 \times M_2 \to (0,\infty)$. Then*

- *$M = M_1 \times_f M_2$ is a warped product with the metric tensor $g = g_1 \oplus f^2 g_2$.*
- *$grad^1 f$ is the gradient of f on M_1 and $\left\|grad^1 f\right\|^2 = g\left(grad^1 f, grad^1 f\right)$.*
- *$gradh$ is the gradient of h on M and $\|gradh\|^2 = g\left(gradh, gradh\right)$.*
- *The same notation is used to denote a vector field and its lift to the sequential warped product manifold.*

We shall denote by $\overline{\nabla}$, ∇^1, ∇^2, ∇^3, $\overline{Ric}$, Ric^1, Ric^2, Ric^3, the Levi-Civita connections and the Ricci curvatures of $\overline{M}$, M_1, M_2 and M_3, respectively.

Now, we give the following lemmas:

Lemma 1.1. *[12] Let $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ be a sequential warped product with metric $\overline{g} = g_1 \oplus f^2 g_2 \oplus h^2 g_3$. If $X_i, Y_i \in \chi\left(M_i\right)$ for $i \in \{1, 2, 3\}$, then*
i) $\overline{\nabla}_{X_1} Y_1 = \nabla^1_{X_1} Y_1$,
ii) $\overline{\nabla}_{X_1} X_2 = \overline{\nabla}_{X_2} X_1 = X_1(lnf)X_2$,
iii) $\overline{\nabla}_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) grad^1 f$,
iv) $\overline{\nabla}_{X_3} X_1 = \overline{\nabla}_{X_1} X_3 = X_1(lnh)X_3$,
v) $\overline{\nabla}_{X_2} X_3 = \overline{\nabla}_{X_3} X_2 = X_2(lnh)X_3$,
vi) $\overline{\nabla}_{X_3} Y_3 = \nabla^3_{X_3} Y_3 - h g_3(X_3, Y_3) gradh$.

Lemma 1.2. *[12] Let $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ be a sequential warped product with metric $\overline{g} = g_1 \oplus f^2 g_2 \oplus h^2 g_3$. If $X_i, Y_i, Z_i \in \chi\left(M_i\right)$, then*
i) $\overline{R}(X_1, Y_1)Z_1 = R^1(X_1, Y_1)Z_1$,
ii) $\overline{R}(X_2, Y_2)Z_2 = R^2(X_2, Y_2)Z_2 - \left\|grad^1 f\right\|^2 \left[g_2\left(X_2, Z_2\right) Y_2 - g_2\left(Y_2, Z_2\right) X_2\right]$,
iii) $\overline{R}(X_1, Y_2)Z_1 = -\frac{1}{f} H_1^f(X_1, Z_1) Y_2$,
iv) $\overline{R}(X_1, Y_2)Z_2 = f g_2\left(Y_2, Z_2\right) \nabla^1_{X_1} grad^1 f$,
v) $\overline{R}(X_1, Y_2)Z_3 = 0$,
vi) $\overline{R}(X_i, Y_i)Z_j = 0$ *for* $i \neq j$,
vii) $\overline{R}(X_i, Y_3)Z_j = -\frac{1}{h} H^h(X_i, Z_j) Y_3$ *for* $i, j = 1, 2$,
viii) $\overline{R}(X_i, Y_3)Z_3 = h g_3\left(Y_3, Z_3\right) \nabla_{X_1} gradh$ *for* $i = 1, 2$,
ix) $\overline{R}(X_3, Y_3)Z_3 = R^3(X_3, Y_3)Z_3 - \|gradh\|^2 \left[g_3\left(X_3, Z_3\right) Y_3 - g_3\left(Y_3, Z_3\right) X_3\right]$.

Lemma 1.3. *[12] Let $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ be a sequential warped product with metric $\overline{g} = g_1 \oplus f^2 g_2 \oplus h^2 g_3$. If $X_i, Y_i \in \chi\left(M_i\right)$, then*
i) $\overline{Ric}(X_1, Y_1) = Ric^1(X_1, Y_1) - \frac{m_2}{f} H_1^f(X_1, Y_1) - \frac{m_3}{h} H^h(X_1, Y_1)$,

ii) $\overline{Ric}(X_2, Y_2) = Ric^2(X_2, Y_2) - \left(f\Delta^1 f + (m_2 - 1)\left\|grad^1 f\right\|^2\right) g_2(X_2, Y_2) - \frac{m_3}{h} H^h(X_2, Y_2)$,

iii) $\overline{Ric}(X_3, Y_3) = Ric^3(X_3, Y_3) - \left(h\Delta h + (m_3 - 1)\|gradh\|^2\right) g_3(X_3, Y_3)$,

iv) $\overline{Ric}(X_i, X_j) = 0$ *for* $i \neq j$.

In [5], Chaki and Maity introduced the notion of quasi-Einstein manifolds. In [7], De and Ghosh studied some properties of quasi Einstein manifolds and considered quasi-Einstein hypersurfaces of Euclidean spaces. In [21], Sular and the second author studied quasi-Einstein warped product manifolds for arbitrary dimension $n \geq 3$. In [14], Dumitru gave the expressions of the Ricci tensors and scalar curvatures for the bases and fibers on quasi-Einstein warped product manifolds. For further developments about quasi-Einstein manifolds; see [1], [8], [10], [15] and [20]. In [12], De, Shenawy and Ünal defined sequential warped product manifolds. In [19], Pahan and Pal studied the Einstein sequential warped product space with negative scalar curvature and gave an example of the Einstein sequential warped space. In [22], Şahin studied sequential warped product submanifolds of Kaehler manifolds and gave some examples. Motivated by the above studies, in the present paper, we consider quasi-Einstein sequential warped product manifolds. Firstly, we obtain that the sequential warped product manifolds are Riemannian products under certain conditions. Furthermore, we investigate the necessary conditions for a sequential standard static space-time and a sequential generalized Robertson-Walker space-time to be a manifold of quasi-constant curvature.

2. Quasi-Einstein sequential warped products

Let $\overline{A}$ be a 1-form on a sequential warped product manifold $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ defined as

$$\overline{g}(X, U) = \overline{A}(X),$$

for any vector field X on $\overline{M}$ and a unit vector field U on $\overline{M}$.

Using Lemma 1.3, we obtain the following corollaries:

Corollary 2.1. *When* $U \in \chi(M_1)$*, the sequential warped product* $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ *is a quasi-Einstein manifold whose Ricci tensor is of the form* $\overline{Ric} = \alpha\overline{g} + \beta\overline{A} \otimes \overline{A}$ *if and only if*

$$Ric^1(X_1, Y_1) = \alpha g_1(X_1, Y_1) + \beta g_1(X_1, U) g_1(Y_1, U) + \frac{m_2}{f} H_1^f(X_1, Y_1) + \frac{m_3}{h} H^h(X_1, Y_1), \tag{2.1}$$

$$Ric^2(X_2, Y_2) = \lambda g_2(X_2, Y_2) + \frac{m_3}{h} H^h(X_2, Y_2), \tag{2.2}$$

where

$$\lambda = \alpha f^2 + f\Delta^1 f + (m_2 - 1)\left\|grad^1 f\right\|^2, \tag{2.3}$$

and

$$Ric^3(X_3, Y_3) = \nu g_3(X_3, Y_3), \tag{2.4}$$

where

$$\nu = \alpha h^2 + h\Delta h + (m_3 - 1)\|gradh\|^2. \tag{2.5}$$

Corollary 2.2. *When* $U \in \chi(M_2)$*, the sequential warped product* $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ *is a quasi-Einstein manifold whose Ricci tensor is of the form* $\overline{Ric} = \alpha\overline{g} + \beta\overline{A} \otimes \overline{A}$ *if and only if*

$$Ric^1(X_1, Y_1) = \alpha g_1(X_1, Y_1) + \frac{m_2}{f} H_1^f(X_1, Y_1) + \frac{m_3}{h} H^h(X_1, Y_1), \tag{2.6}$$

$$Ric^2(X_2, Y_2) = \lambda g_2(X_2, Y_2) + \beta f^4 g_2(X_2, U) g_2(Y_2, U) + \frac{m_3}{h} H^h(X_2, Y_2), \tag{2.7}$$

where

$$\lambda = \alpha f^2 + f\Delta^1 f + (m_2 - 1)\left\|grad^1 f\right\|^2, \tag{2.8}$$

and

$$Ric^3(X_3, Y_3) = \nu g_3(X_3, Y_3), \tag{2.9}$$

where

$$\nu = \alpha h^2 + h\Delta h + (m_3 - 1)\|gradh\|^2. \tag{2.10}$$

Corollary 2.3. *When $U \in \chi(M_3)$, the sequential warped product $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ is a quasi-Einstein manifold whose Ricci tensor is of the form $\overline{Ric} = \alpha\overline{g} + \beta\overline{A} \otimes \overline{A}$ if and only if*

$$Ric^1(X_1, Y_1) = \alpha g_1(X_1, Y_1) + \frac{m_2}{f} H_1^f(X_1, Y_1) + \frac{m_3}{h} H^h(X_1, Y_1), \tag{2.11}$$

$$Ric^2(X_2, Y_2) = \lambda g_2(X_2, Y_2) + \frac{m_3}{h} H^h(X_2, Y_2), \tag{2.12}$$

where

$$\lambda = \alpha f^2 + f\Delta^1 f + (m_2 - 1)\left\|grad^1 f\right\|^2, \tag{2.13}$$

and

$$Ric^3(X_3, Y_3) = \nu g_3(X_3, Y_3) + \beta h^4 g_3(X_3, U) g_3(Y_3, U), \tag{2.14}$$

where

$$\nu = \alpha h^2 + h\Delta h + (m_3 - 1)\|gradh\|^2. \tag{2.15}$$

Lemma 2.1. *[17] Let f be a smooth function on M_1. Then, the divergence of the Hessian tensor H^f satisfies*

$$div\left(H^f\right)(X) = Ric(gradf, X) - \Delta^1(df)(X),$$

where Δ^1 denotes the Laplacian on M_1.

Lemma 2.2. *[19] Let h be a smooth function on $M_1 \times M_2$. Then, the divergence of the Hessian tensor H^h satisfies*

$$div\left(H^h\right)(X) = Ric(gradh, X) - \Delta(dh)(X),$$

where Δ denotes the Laplacian on $M_1 \times M_2$.

Using Lemma 2.1 and Lemma 2.2, we can state the following proposition for $U \in \chi(M_1)$:

Proposition 2.1. *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds with $m_1 \geq 2$ and $m_2 \geq 2$.*

Suppose that f is a non-constant smooth function on M_1 satisfying

$$Ric^1(X_1, Y_1) = \alpha g_1(X_1, Y_1) + \beta g_1(X_1, U) g_1(Y_1, U) + \frac{m_2}{f} H_1^f(X_1, Y_1) + \frac{m_3}{h} H^h(X_1, Y_1), \tag{2.16}$$

for constants α, β and $U \in \chi(M_1)$. If the condition

$$\begin{aligned} &\frac{m_2\beta}{f} g_1(grad^1 f, U) g_1(X, U) + \frac{m_2 m_3}{fh} H^h(grad^1 f, X) + m_3 div(\frac{H^h}{h})(X) \\ &= \frac{m_3}{2} d(\frac{\Delta h}{h})(X) + \frac{2m_2}{f} d\left(\Delta^1 f\right)(X) \end{aligned} \tag{2.17}$$

holds, then f satisfies

$$\lambda = \alpha f^2 + f\Delta^1 f + (m_2 - 1)\left\|grad^1 f\right\|^2 \tag{2.18}$$

for a constant λ.

Suppose that h is a non-constant smooth function on $M_1 \times M_2$ satisfying

$$Ric^2(X_2, Y_2) = \lambda g_2(X_2, Y_2) + \frac{m_3}{h} H^h(X_2, Y_2) \tag{2.19}$$

for a constant λ and $U \in \chi(M_1)$. If the condition

$$(\lambda - \alpha)dh = d(\Delta h) \tag{2.20}$$

holds, then h satisfies

$$\nu = \alpha h^2 + h\Delta h + (m_3 - 1)\|gradh\|^2 \tag{2.21}$$

for a constant ν. Hence, for a compact Einstein space (M_3, g_3) of dimension $m_3 \geq 2$ such that the Ricci tensor of (M_3, g_3) is of the form $Ric^3 = \nu g_3$, we can make a quasi-Einstein sequential warped product space $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ whose Ricci tensor is of the form $\overline{Ric} = \alpha\overline{g} + \beta\overline{A} \otimes \overline{A}$.

Proof. Assume that $U \in \chi(M_1)$. By taking trace of both sides of (2.16), we have

$$scal_1 = \alpha m_1 + \beta + \frac{m_2}{f}\Delta^1 f + \frac{m_3}{h}\Delta h. \tag{2.22}$$

By the use of the second Bianchi identity, we have

$$d(scal_1) = 2div(Ric^1). \tag{2.23}$$

From equations (2.22) and (2.23), we get

$$div(Ric^1) = \frac{m_2}{2f^2}\left[-\left(\Delta^1 f\right)(df) + fd\left(\Delta^1 f\right)\right] + \frac{m_3}{2h^2}\left[-(\Delta h)(dh) + hd(\Delta h)\right]. \tag{2.24}$$

By the definition of divergence, we have

$$div(\frac{1}{f}H_1^f)(X) = -\frac{1}{f^2}H_1^f(grad^1 f, X) + \frac{1}{f}divH_1^f(X) \tag{2.25}$$

for any vector field X on M_1. Since $H_1^f(grad^1 f, X) = \frac{1}{2}d(\|grad^1 f\|^2)(X)$, equation (2.25) turns into

$$div(\frac{1}{f}H_1^f)(X) = -\frac{1}{2f^2}d(\|grad^1 f\|^2)(X) + \frac{1}{f}divH_1^f(X). \tag{2.26}$$

Using Lemma 2.1 and equation (2.1) in (2.26), we find

$$\begin{aligned} div(\frac{1}{f}H_1^f)(X) &= \frac{1}{2f^2}\left[(m_2-1)d(\|grad^1 f\|^2)(X) + (2\alpha f)df(X) - (2f)d\left(\Delta^1 f\right)(X)\right] \\ &+\frac{\beta}{f}g_1(grad^1 f, U)g_1(X,U) + \frac{m_3}{fh}H^h(grad^1 f, X). \end{aligned} \tag{2.27}$$

From equation (2.1), we can write

$$div(Ric^1)(X) = m_2 div(\frac{1}{f}H_1^f)(X) + m_3 div(\frac{1}{h}H^h)(X). \tag{2.28}$$

Using equation (2.27) into (2.28), we obtain

$$\begin{aligned} div(Ric^1)(X) &= \frac{m_2}{2f^2}\left[(m_2-1)d(\|grad^1 f\|^2)(X) + (2\alpha f)df(X) - (2f)d\left(\Delta^1 f\right)(X)\right] \\ &+m_2\frac{\beta}{f}g_1(grad^1 f, U)g_1(X,U) + \frac{m_2 m_3}{fh}H^h(grad^1 f, X) + m_3 div(\frac{1}{h}H^h)(X). \end{aligned} \tag{2.29}$$

From equations (2.24) and (2.29), we get

$$\begin{aligned} &\frac{m_2}{2f^2}d\left[(m_2-1)(\|grad^1 f\|^2) + (\alpha f^2) + f\Delta^1 f\right](X) \\ &+m_2\frac{\beta}{f}g_1(grad^1 f, U)g_1(X,U) - \frac{2m_2}{f}d(\Delta^1 f)(X) + \frac{m_2 m_3}{fh}H^h(grad^1 f, X) \\ &+m_3 div(\frac{1}{h}H^h)(X) - \frac{m_3}{2}d\left(\frac{\Delta h}{h}\right)(X) = 0. \end{aligned} \tag{2.30}$$

Using the condition (2.17), we have

$$d\left[(m_2-1)(\|grad^1 f\|^2) + \alpha f^2 + f\Delta^1 f\right](X) = 0.$$

Therefore, we obtain (2.18). Thus, the first part of the proposition is proved.

Similarly, by taking the trace of both sides of equation (2.19), we get

$$scal_2 = \lambda m_2 + m_3\frac{\Delta h}{h}. \tag{2.31}$$

Similarly, from Lemma 2.2, we have equation (2.21) for a constant ν, if the condition (2.20) holds. Thus, the second part of the proposition is proved. For a compact Einstein manifold (M_3, g_3) of dimension $m_3 \geq 2$ with $Ric^3 = \nu g_3$, we can generate a quasi-Einstein sequential warped product $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ whose Ricci tensor is of the form $\overline{Ric} = \alpha\overline{g} + \beta\overline{A}\otimes\overline{A}$, by using Corollary 2.1 for $U \in \chi(M_1)$. This completes the proof. □

Using Lemma 2.1, Lemma 2.2 and Corollary 2.2, we have the following proposition for $U \in \chi(M_2)$:

Proposition 2.2. *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds with $m_1 \geq 2$ and $m_2 \geq 2$.*

Suppose that f is a non-constant smooth function on M_1 satisfying

$$Ric^1(X_1, Y_1) = \alpha g_1(X_1, Y_1) + \frac{m_2}{f} H_1^f(X_1, Y_1) + \frac{m_3}{h} H^h(X_1, Y_1),$$

for a constant α and $U \in \chi(M_2)$. If the condition

$$\frac{m_2 m_3}{fh} H^h(grad^1 f, X) + m_3 div(\frac{H^h}{h})(X) = \frac{m_3}{2} d(\frac{\Delta h}{h})(X) + \frac{2m_2}{f} d\left(\Delta^1 f\right)(X)$$

holds, then f satisfies

$$\lambda = \alpha f^2 + f\Delta^1 f + (m_2 - 1)\left\|grad^1 f\right\|^2 \tag{2.32}$$

for a constant λ.

Suppose that h is a non-constant smooth function on $M_1 \times M_2$ satisfying

$$Ric^2(X_2, Y_2) = \lambda g_2(X_2, Y_2) + \beta f^4 g_2(X_2, U) g_2(Y_2, U) + \frac{m_3}{h} H^h(X_2, Y_2)$$

for a constant λ and $U \in \chi(M_2)$. If the condition

$$\frac{m_3}{h}(\lambda - \alpha) dh(X) + \beta div(f^4)(X) + \frac{m_3\beta}{h} f^4 g_2(gradh, U) g_2(X, U) = \frac{2m_3}{h} d(\Delta h)(X) + 2\beta f^3 df(X)$$

holds, then h satisfies

$$\nu = \alpha h^2 + h\Delta h + (m_3 - 1)\|gradh\|^2 \tag{2.33}$$

for a constant ν. Hence, for a compact Einstein space (M_3, g_3) of dimension $m_3 \geq 2$ such that the Ricci tensor of (M_3, g_3) is of the form $Ric^3 = \nu g_3$, we can make a quasi-Einstein sequential warped product space $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ whose Ricci tensor is of the form $\overline{Ric} = \alpha \overline{g} + \beta \overline{A} \otimes \overline{A}$.

Proof. The proof is similar to the proof of Proposition 2.1. □

Using Lemma 2.1, Lemma 2.2 and Corollary 2.3, we have the following proposition for $U \in \chi(M_3)$:

Proposition 2.3. *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds with $m_1 \geq 2$ and $m_2 \geq 2$.*

Suppose that f is a non-constant smooth function on M_1 satisfying

$$Ric^1(X_1, Y_1) = \alpha g_1(X_1, Y_1) + \frac{m_2}{f} H_1^f(X_1, Y_1) + \frac{m_3}{h} H^h(X_1, Y_1)$$

for a constant α and $U \in \chi(M_3)$. If the condition

$$\frac{m_2 m_3}{fh} H^h(grad^1 f, X) + m_3 div(\frac{H^h}{h})(X) = \frac{m_3}{2} d(\frac{\Delta h}{h})(X) + \frac{2m_2}{f} d\left(\Delta^1 f\right)(X)$$

holds, then f satisfies

$$\lambda = \alpha f^2 + f\Delta^1 f + (m_2 - 1)\left\|grad^1 f\right\|^2 \tag{2.34}$$

for a constant λ.

Suppose that h is a non-constant smooth function on $M_1 \times M_2$ satisfying

$$Ric^2(X_2, Y_2) = \lambda g_2(X_2, Y_2) + \frac{m_3}{h} H^h(X_2, Y_2)$$

for a constant λ and $U \in \chi(M_3)$. If the condition

$$(\lambda - \alpha) dh = d(\Delta h)$$

holds, then h satisfies

$$\nu = \alpha h^2 + h\Delta h + (m_3 - 1)\|gradh\|^2 \tag{2.35}$$

for a constant ν. Hence, for a compact quasi-Einstein space (M_3, g_3) of dimension $m_3 \geq 2$ such that the Ricci tensor of (M_3, g_3) is of the form $Ric^3 = \nu g_3 + \beta_1 A_3 \otimes A_3$, where A_3 is a non-zero 1-form of M_3, we can make a quasi-Einstein sequential warped product space $\overline{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ whose Ricci tensor is of the form $\overline{Ric} = \alpha \overline{g} + \beta \overline{A} \otimes \overline{A}$.

Proof. The proof is similar to the proof of Proposition 2.1. □

Now, we can state the following theorem:

Theorem 2.1. *Let $\overline{M}=\left(M_1\times_f M_2\right)\times_h M_3$ be a compact quasi-Einstein sequential warped product space whose Ricci tensor is of the form $\overline{Ric}=\alpha\overline{g}+\beta\overline{A}\otimes\overline{A}$. If α is a negative constant, then the sequential warped product becomes a Riemannian product.*

Proof. The equation (2.3) gives us

$$\lambda=\alpha f^2+div(f\Delta^1 f)+(m_2-2)\left\|grad^1 f\right\|^2. \tag{2.36}$$

By integrating over M_1, we have

$$\lambda=\frac{\alpha}{\vartheta\left(M_1\right)}\int\limits_{M_1} f^2+\frac{(m_2-2)}{\vartheta\left(M_1\right)}\int\limits_{M_1}\left\|grad^1 f\right\|^2, \tag{2.37}$$

where $\vartheta\left(M_1\right)$ denotes the volume of M_1.

Case I: Assume that $m_2\geq 3$. Let p be a maximum point of f on M_1. Hence, we have $f(p)>0$, $grad^1 f(p)=0$ and $\Delta^1 f(p)\geq 0$. Using equations (2.3) and (2.37), we can write

$$\begin{aligned}
0 &\leq f\left(p\right)\Delta^1 f\left(p\right)\\
&=\alpha f(p)^2-\lambda\\
&=\frac{(2-m_2)}{\vartheta\left(M_1\right)}\int\limits_{M_1}\left\|grad^1 f\right\|^2+\frac{\alpha}{\vartheta\left(M_1\right)}\int\limits_{M_1}\left(f(p)^2-f^2\right)\leq 0.
\end{aligned}$$

If α is a negative constant, then f is a constant.

Case II: Assume that $m_2=2$. Let q be a minimum point of f on M_1. Hence, we have $f(q)>0$, $grad^1 f(q)=0$ and $\Delta^1 f(q)\leq 0$. Thus, we can write

$$\begin{aligned}
0 &\geq f\left(q\right)\Delta^1 f\left(q\right)\\
&=\alpha f(q)^2-\lambda\\
&=\frac{(2-m_2)}{\vartheta\left(M_1\right)}\int\limits_{M_1}\left\|grad^1 f\right\|^2+\frac{\alpha}{\vartheta\left(M_1\right)}\int\limits_{M_1}\left(f(q)^2-f^2\right)\geq 0.
\end{aligned}$$

If α is a negative constant, then f is a constant.

Similarly, equation (2.5) gives us

$$\nu=\alpha h^2+div(h\Delta h)+(m_3-2)\left\|gradh\right\|^2. \tag{2.38}$$

By integrating over $M_1\times M_2$, we have

$$\nu=\frac{\alpha}{\vartheta\left(M_1\times M_2\right)}\int\limits_{M_1\times M_2} h^2+\frac{(m_3-2)}{\vartheta\left(M_1\times M_2\right)}\int\limits_{M_1\times M_2}\left\|gradh\right\|^2, \tag{2.39}$$

where $\vartheta\left(M_1\times M_2\right)$ denotes the volume of $M_1\times M_2$.

Case I: Assume that $m_3\geq 3$. Let (p_1,p_2) be a maximum point of h on $M_1\times M_2$. Hence, we have $h\left(p_1,p_2\right)>0$, $gradh\left(p_1,p_2\right)=0$ and $\Delta h\left(p_1,p_2\right)\geq 0$. Using equations (2.5) and (2.39), we get

$$\begin{aligned}
0 &\leq h\left(p_1,p_2\right)\Delta h\left(p_1,p_2\right)\\
&=\alpha h\left(p_1,p_2\right)^2-\nu\\
&=\frac{(2-m_3)}{\vartheta\left(M_1\times M_2\right)}\int\limits_{M_1\times M_2}\left\|gradh\right\|^2+\frac{\alpha}{\vartheta\left(M_1\times M_2\right)}\int\limits_{M_1\times M_2}\left(h\left(p_1,p_2\right)^2-h^2\right)\leq 0.
\end{aligned}$$

If α is a negative constant, then h is a constant.

Case II: Assume that $m_3=2$. Let (q_1,q_2) be a minimum point of h on $M_1\times M_2$. Hence, we have $h\left(q_1,q_2\right)>0$, $gradh\left(q_1,q_2\right)=0$ and $\Delta h\left(q_1,q_2\right)\leq 0$. Thus, we get

$$\begin{aligned}0 &\geq h(q_1,q_2)\,\Delta h(q_1,q_2)\\ &= \alpha h(q_1,q_2)^2 - \nu\\ &= \frac{(2-m_3)}{\vartheta(M_1\times M_2)}\int\limits_{M_1\times M_2}\|gradh\|^2 + \frac{\alpha}{\vartheta(M_1\times M_2)}\int\limits_{M_1\times M_2}\left(h(q_1,q_2)^2 - h^2\right)\geq 0.\end{aligned}$$

If α is a negative constant, then h is a constant. Hence, the sequential warped product becomes a Riemannian product. Similarly, same results are obtained for equations (2.8) and (2.13), (2.10) and (2.15). This completes the proof. □

Using Corollary 2.1, Corollary 2.2 and Corollary 2.3, we have the following theorem for $U\in\chi(M_i)$ for $1\leq i\leq 3$:

Theorem 2.2. *Let $\overline{M}=\left(M_1\times_f M_2\right)\times_h M_3$ be a quasi-Einstein sequential warped product space, where M_1, M_2 and M_3 are three compact spaces. Then, the following conditions hold:*

i) a) For $U\in\chi(M_1)$ or $U\in\chi(M_2)$, if $scal_3\leq 0$, $\alpha\geq 0$ and $\Delta h\geq 0$, then h is a constant.

b) For $U\in\chi(M_3)$, if $scal_3\leq 0$, $\alpha,\beta\geq 0$ and $\Delta h\geq 0$, then h is a constant.

ii) a) For $U\in\chi(M_1)$ or $U\in\chi(M_3)$, if $m_2=1$ and (either $\lambda\geq\alpha f^2$ or $\lambda\leq\alpha f^2$), then f is a constant. Moreover, h is a constant if $\alpha\geq 0$ and $\Delta h\geq 0$. Hence, the sequential warped product is a Riemannian product.

b) For $U\in\chi(M_2)$, if $m_2=1$ and (either $\lambda\geq\alpha f^2$ or $\lambda\leq\alpha f^2$), then f is a constant. Moreover, h is a constant if $\alpha,\beta\geq 0$ and $\Delta h\geq 0$. Hence, the sequential warped product is a Riemannian product.

iii) For $U\in\chi(M_i)$, $1\leq i\leq 3$, if $\left\|grad^1 f\right\|\geq\sqrt{\frac{\lambda}{m_2-1}}$, $\|gradh\|\geq\sqrt{\frac{\nu}{m_3-1}}$, $\alpha\leq 0$, $\Delta^1 f\leq 0$ and $\Delta h\leq 0$, then f and h are constants. Hence, the sequential warped product is a Riemannian product.

Proof. Assume that $U\in\chi(M_1)$. Taking the trace of Corollary 2.1, we have

$$scal_1 = \alpha m_1 + \beta + \frac{m_2}{f}\Delta^1 f + \frac{m_3}{h}\Delta h, \tag{2.40}$$

$$scal_2 = \lambda m_2 + \frac{m_3}{h}\Delta h \tag{2.41}$$

and

$$scal_3 = \nu m_3. \tag{2.42}$$

i) Using equation (2.42), if $scal_3\leq 0$, then $\nu\leq 0$. From Corollary 2.1, we can write

$$\alpha h^2 + h\Delta h = \nu - (m_3-1)\|gradh\|^2 \leq 0.$$

If $\alpha\geq 0$ and $\Delta h\geq 0$, then we have

$$0\leq h\Delta h\leq -\alpha h^2\leq 0.$$

Therefore, h is a constant.

ii) From Corollary 2.1 and $m_2=1$, we have

$$\lambda - \alpha f^2 = f\Delta^1 f.$$

Using the above equation, if either $\lambda\geq\alpha f^2$ or $\lambda\leq\alpha f^2$, then f is a constant. So, we can write

$$\lambda = \alpha f^2. \tag{2.43}$$

Using (2.41), (2.43) and $m_2=1$, we get

$$\alpha f^2 + \frac{m_3}{h}\Delta h = 0. \tag{2.44}$$

From equation (2.44), if $\alpha\geq 0$ and $\Delta h\geq 0$, then h is a constant.

iii) From Corollary 2.1, we have

$$\alpha f^2 + f\Delta^1 f = \lambda - (m_2-1)\left\|grad^1 f\right\|^2 \tag{2.45}$$

and

$$\alpha h^2 + h\Delta h = \nu - (m_3-1)\|gradh\|^2. \tag{2.46}$$

Using the conditions $\|grad^1 f\| \geq \sqrt{\frac{\lambda}{m_2-1}}$, $\|gradh\| \geq \sqrt{\frac{\nu}{m_3-1}}$, $\alpha \leq 0$, $\Delta^1 f \leq 0$ and $\Delta h \leq 0$ for equations (2.45) and (2.46), respectively, we get

$$0 \leq -\alpha f^2 \leq f\Delta^1 f \leq 0$$

and

$$0 \leq -\alpha h^2 \leq h\Delta h \leq 0.$$

Therefore, we deduce that f and h are constants. The proofs for $U \in \chi(M_2)$ and $U \in \chi(M_3)$ are similar. This completes the proof. □

3. Applications

Warped products have attracted great attention in differential geometry and physics, since the exact solutions of Einstein's equation are obtained with the help of warped product space-time models [2], [3]. Generalized Robertson-Walker space-times and standard static space-times are two well known solutions of Einstein's field equations. The standard static space-times can be considered as a generalization of the Einstein static universe [13]. In addition, the Robertson-Walker models in general relativity describe a simply connected homogeneous isotropic expanding or contracting universe. Then, generalized Robertson-Walker space-times extend the Robertson-Walker space-times [9]. Sequential generalized Robertson-Walker and sequential standard static space-times are introduced in [12].

Let (M_i, g_i) be m_i-dimensional Riemannian manifolds, where $i \in \{1, 2, 3\}$. Let $f : M_1 \to (0, \infty)$ and $h : M_1 \times M_2 \to (0, \infty)$ be two smooth positive functions. Then $(m_1 + m_2 + 1)$-dimensional product manifold $\overline{M} = \left(M_1 \times_f M_2\right) \times_h I$ equipped with the metric tensor

$$\overline{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2(-dt^2) \tag{3.1}$$

is a sequential standard static space-time, where I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the Euclidean metric tensor on I [12]. In addition, $(m_2 + m_3 + 1)$-dimensional product manifold $\overline{M} = \left(I \times_f M_2\right) \times_h M_3$ equipped with the metric tensor

$$\overline{g} = \left(-dt^2 \oplus f^2 g_2\right) \oplus h^2 g_3 \tag{3.2}$$

is a sequential generalized Robertson-Walker space-time [12].

Theorem 3.1. *Let $\overline{M} = \left(M_1 \times_f M_2\right) \times_h I$ be a sequential standard static space-time with metric $\overline{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2(-dt^2)$. If $\left(\overline{M}, \overline{g}\right)$ is a manifold of quasi-constant curvature, then the following conditions are satisfied with $a, b \in C^\infty\left(\overline{M}, \mathbb{R}\right)$ and $b \neq 0$:*

i) $(m_1 + m_2)a + b = (m_1 + m_2 - 1)bh^2 - \frac{\Delta h}{h}$,

ii) (M_1, g_1) is a quasi-Einstein manifold such that the Hessian tensors H_1^f and H^h satisfy the following expressions with $U = U_1 + \partial_t$

$$H_1^f(X_1, Y_1) = afg_1(X_1, Y_1) + bfg_1(X_1, U_1)g_1(Y_1, U_1)$$

and

$$H^h(X_1, Y_1) = (-a + h^2)hg_1(X_1, Y_1) - bhg_1(X_1, U_1)g_1(Y_1, U_1),$$

iii) (M_2, g_2) is an Einstein manifold such that the Hessian tensor H^h satisfies the following expression with $U = U_1 + \partial_t$

$$H^h(X_2, Y_2) = (-a + h^2)f^2 hg_2(X_2, Y_2).$$

Proof. From [12], we have

$$\overline{Ric}(X_1, Y_1) = Ric^1(X_1, Y_1) - \frac{m_2}{f}H_1^f(X_1, Y_1) - \frac{1}{h}H^h(X_1, Y_1), \tag{3.3}$$

$$\overline{Ric}(X_2, Y_2) = Ric^2(X_2, Y_2)$$
$$-\left(f\Delta^1 f + (m_2 - 1)\left\|grad^1 f\right\|^2\right)g_2(X_2, Y_2) - \frac{1}{h}H^h(X_2, Y_2), \tag{3.4}$$

$$\overline{Ric}(\partial_t, \partial_t) = h\Delta h. \tag{3.5}$$

Let $\left(\overline{M}, \overline{g}\right)$ be a manifold of quasi-constant curvature. Thus, $\left(\overline{M}, \overline{g}\right)$ is a quasi-Einstein manifold. Using equations (1.1) and (1.2), we have

$$\overline{Ric}(\partial_t, \partial_t) = -\alpha h^2 + \beta h^4. \tag{3.6}$$

From equations (3.5) and (3.6), we find

$$\alpha = \beta h^2 - \frac{\Delta h}{h}. \tag{3.7}$$

Using (1.1) and (1.3), we can write

$$\alpha = (m_1 + m_2)\, a + b \tag{3.8}$$

and

$$\beta = (m_1 + m_2 - 1)\, b. \tag{3.9}$$

Substituting equations (3.8) and (3.9) in (3.7), we obtain (i).

By Theorem 4.8 in [12], we have

$$H^h(X_1, Y_1) = (-a + h^2) h g_1(X_1, Y_1) - bh g_1(X_1, U_1) g_1(Y_1, U_1) \tag{3.10}$$

and

$$H_1^f(X_1, Y_1) = af g_1(X_1, Y_1) + bf g_1(X_1, U_1) g_1(Y_1, U_1) \tag{3.11}$$

with $U = U_1 + \partial_t$. By the use of equations (1.1), (3.10) and (3.11) into (3.3), we find

$$Ric^1(X_1, Y_1) = (\alpha + (m_2 - 1)\, a + h^2) g_1(X_1, Y_1) + (\beta + (m_2 - 1)\, b)\, g_1(X_1, U_1) g_1(Y_1, U_1).$$

Hence, (M_1, g_1) is a quasi-Einstein manifold.

From Theorem 4.8 in [12], we find

$$H^h(X_2, Y_2) = (-a + h^2) f^2 h g_2(X_2, Y_2), \tag{3.12}$$

with $U = U_1 + \partial_t$. Using equations (1.1) and (3.12) into (3.4), we get

$$Ric^2(X_2, Y_2) = \left(\alpha f^2 + f \Delta^1 f + (m_2 - 1) \left\| grad^1 f \right\|^2 - af^2 + f^2 h^2\right) g_2(X_2, Y_2).$$

Thus, (M_2, g_2) is an Einstein manifold. This completes the proof. □

Theorem 3.2. *Let* $\overline{M} = \left(I \times_f M_2\right) \times_h M_3$ *be a sequential generalized Robertson-Walker space-time with metric* $\overline{g} = \left(-dt^2 \oplus f^2 g_2\right) \oplus h^2 g_3$. *If* $\left(\overline{M}, \overline{g}\right)$ *is a manifold of quasi-constant curvature, then the following conditions are satisfied with* $a, b \in C^\infty\left(\overline{M}, \mathbb{R}\right)$ *and* $b \neq 0$:

i) $(m_2 + m_3 - 2)\, b - (m_2 + m_3)\, a = \frac{m_2}{f} f'' + \frac{m_3}{h} \frac{\partial^2 h}{\partial t^2}$,

ii) (M_2, g_2) *is a quasi-Einstein manifold such that the Hessian tensor* H^h *satisfies the following expression with* $U = \partial_t + U_2$,

$$H^h(X_2, Y_2) = ahf^2 g_2(X_2, Y_2) + bhf^4 g_2(X_2, U_2) g_2(Y_2, U_2),$$

iii) (M_3, g_3) *is an Einstein manifold with* $U = \partial_t + U_2$.

Proof. From [12], we have

$$\overline{Ric}(\partial_t, \partial_t) = \frac{m_2}{f} f'' + \frac{m_3}{h} \frac{\partial^2 h}{\partial t^2}, \tag{3.13}$$

$$\overline{Ric}(X_2, Y_2) = Ric^2(X_2, Y_2) - \left(-ff'' - (m_2 - 1) \left(f'\right)^2\right) g_2(X_2, Y_2) - \frac{m_3}{h} H^h(X_2, Y_2), \tag{3.14}$$

$$\overline{Ric}(X_3, Y_3) = Ric^3(X_3, Y_3) - \left(h\Delta h + (m_3 - 1) \left\| gradh \right\|^2\right) g_3(X_3, Y_3). \tag{3.15}$$

Let $\left(\overline{M}, \overline{g}\right)$ be a manifold of quasi-constant curvature. Thus, $\left(\overline{M}, \overline{g}\right)$ is a quasi-Einstein manifold. Using equations (1.1) and (1.2), we get

$$\overline{Ric}(\partial_t, \partial_t) = -\alpha + \beta. \tag{3.16}$$

From equations (3.13) and (3.16), we have

$$\beta - \alpha = \frac{m_2}{f} f'' + \frac{m_3}{h} \frac{\partial^2 h}{\partial t^2}. \tag{3.17}$$

By equations (1.1) and (1.3), we can write

$$\alpha = (m_2 + m_3)\, a + b \tag{3.18}$$

and

$$\beta = (m_2 + m_3 - 1)\, b. \tag{3.19}$$

Substituting equations (3.18) and (3.19) in (3.17), we obtain (i).

Using Proposition 4.2 in [12], we have

$$H^h(X_2, Y_2) = ahf^2 g_2(X_2, Y_2) + bhf^4 g_2(X_2, U_2) g_2(Y_2, U_2) \tag{3.20}$$

with $U = \partial_t + U_2$. In view of equations (1.1) and (3.20) into (3.14), it follows that

$$Ric^2(X_2, Y_2) = \left(((m_2 + m_3)\, a + b)\, f^2 - ff'' - (m_2 - 1)\left(f'\right)^2 + m_3 af^2\right) g_2(X_2, Y_2)$$
$$+ (((m_2 + m_3 - 1)\, b) + m_3 b)\, f^4 g_2(X_2, U_2) g_2(Y_2, U_2).$$

Hence, (M_2, g_2) is a quasi-Einstein manifold. Using equations (1.1) into (3.15), we find

$$Ric^3(X_3, Y_3) = \left(((m_2 + m_3)\, a + b)\, h^2 + h\Delta h + (m_3 - 1)\, \|gradh\|^2\right) g_3(X_3, Y_3).$$

Thus, (M_3, g_3) is an Einstein manifold with $U = \partial_t + U_2$. This completes the proof. □

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