Journal of Science and Technology
3 (2), 2009, 301 – 319
©BEYKENT UNIVERSITY

($^n1b_1$, $^n2b_2$, $^n3b_3$) Linear Codes Over GF(2)

Vinod TYAGI[1] and Amita SETHI[2]
[1]Department of Mathematics Shyamlal College (Eve.) University of Delhi, Shahdara, Delhi 110032 INDIA
vinodtyagi@hotmail.com
[2]Research Scholar, Department of Mathematics, University of Delhi
Delhi-110007 INDIA,
amita_sethi_23@indiatimes.com

Received: 25.09.2009, Revised: 02.10.2009, Accepted: 20.11.2009

Abstract

This paper explores the possibilities of the existence of block-wise burst error correcting (n,k) linear codes over GF(2) (Galois field of two elements ,0 and 1) that can correct all bursts of length b_1 (fixed) in the first n_1 components , all bursts of length b_2 (fixed) in the next n_2 components and all bursts of length b_3 (fixed) in the last n_3 components; $n = n_1 + n_2 + n_3$.Such codes are named as ($^n1b_1$, $^n2b_2$, $^n3b_3$) linear codes. Some of these codes turn out to be byte oriented[7].

Key Words *: ($^n1b_1$, $^n2b_2$, $^n3b_3$) code, burst of length b(fixed), parity check matrix, error pattern syndrome- table,byte oriented codes.*

Özet

GF(2) Alanında ($^n1b_1$, $^n2b_2$, $^n3b_3$) Doğrusal Kodları

Bu makale GF(2) (iki elemanlı Galois alanı, 0 ve 1) alanında (n,k) doğrusal kodlarını düzelten blok-akıllı bozulma hatalarının varlığının olasılıklarını ele almaktadır. Doğrusal kodlar ilk n_1 bileşenlerinde b_1 (sabitlenmiş) uzunluğunun bütün bozulmalarını, sonraki n_2 bileşenlerinde b_2 (sabitlenmiş) uzunluğunun bütün bozulmalarını ve en son n_3 bileşenlerindeki b_3 (sabitlenmiş) uzunluğunun bütün bozulmalarını düzeltmektedir, $n_1 + n_2 + n_3 = n$. Bu tür kodlar ($^n1b_1$, $^n2b_2$, $^n3b_3$) doğrusal kodları olarak isimlendirilir. Bu kodlardan bazıları byte işlemlerine uygulanma sonucunu verir [7].

Anahtar sözcükler: *($^n1b_1$, $^n2b_2$, $^n3b_3$) kodu, b sabitlenmiş uzunluğunun bozulması, parite denetim matrisi, hata pattern sendrom tablosu, byte işlemli kodlar.*

INTRODUCTION

In many memory systems, the information is stored in different parts (sub-blocks) of the code length and it is natural to expect errors in these parts during communication .We, therefore partition the code length into various sub-blocks (parts) also referred to as bytes [7],[8], in such a way that the pattern of errors in each sub-block is known. Therefore, when we consider error correction in such a system, we correct errors, which occur in the same sub-block. This arrangement results into fast and efficient detection and correction of errors and many studies have been made in this direction. Hence, there is need to study block-wise error correcting codes (BEC) codes and their error correcting capabilities systematically.

Burst is the most common error in many communications systems and burst error correcting codes are developed to deal with such errors. As we have said earlier, the design of block-wise error correcting (BEC) codes suggest that a burst can occur only with in the selected parts of the code length or sub-blocks. In case of two sub-blocks, Dass,Iembo and Jain ; [5], [6] and Tyagi and Rana [9] ,[10] and others have published some interesting results for binary and non-binary cases. Recently Tuvi Etzion [7],[8] has studied such perfect c byte oriented burst codes where code length is divided into

(1) Bytes of same size;
(2) One byte of size n1 and remaining bytes are of size n2;
(3) Each byte is of size either n1or n2;
(4) The size of each byte is a power of 2;
(5) All other cases.

and the error can occur only within the bytes. For example, in (1), if n is the code length and r is the size of a byte then a code has n/r bytes in total. In this way such byte oriented codes becomes a special case of block wise error correcting codes where sub-blocks are of equal size and so is the burst error. *In an (n, k) linear code, a burst of length b or less is a vector whose all the non-zero components are confined to some b-consecutive positions, the first and the last of which is non-zero.*

In many other such communication systems, contrary to the usual burst error (as defined above), it has been observed that errors do occur but not near the end of the code vectors. Continuous occurrence of this situation has brought the definition of burst due to Chien and Tang [2] with a modification due to Dass [3] into active consideration. According to this definition, *'A burst of length b is a vector whose all the non-zero components are confined to some b consecutive positions, the first of which is non zero and the number of its starting positions in an n-tuple is the first (n-b+1) positions'. It may be noted that according to this definition,(1000000) will be considered as a burst of length up to 7,whereas (0001000)will be a burst of length at most 4.* This definition has been found very useful in error analysis experiments on telephone lines [1] and in communication channels where error normally do not occur near the end of a vector, particularly when burst length is large.

Such b(fixed) BEC burst codes were introduced by Dass and Tyagi [4] in 1980. The authors studied linear codes that could correct two different burst of lengths b_1 (fixed) and b_2 (fixed) in two sub-blocks of length n_1 and n_2 in a code of length n; $n = n_1 + n_2$. An $(n = n_1 + n_2, k)$ code was taken as a subspace of all n-tuples over GF (q). The weight of a vector and the distance between two vectors is considered in the Hamming sense.

In this communication, we divide the code length n into three sub-blocks in order to find the possibilities of the existence of $({}^{n}1b_1, {}^{n}2b_2, {}^{n}3b_3)$ burst error correcting codes. We would also be interested in finding special situations, which may lead to some **new byte correcting codes** with respect to this new definition of burst. The paper is organized into two sections. In section 1, we obtain a necessary condition on the number of parity check digits for codes that can correct bursts of different lengths in three different sub-blocks. In section 2, we obtain a sufficient condition on the number of parity check digits required for constructing a code by following the technique given by Dass and Tyagi [4] where the required parity check matrix was obtained by applying a transformation to the columns of another matrix constructed under suitable constraints. The paper is concluded by giving appropriate examples in the end.

A Necessary Condition

We first obtain a lower bound on the necessary number of parity check digits required for a $({}^{n}1b_1, {}^{n}2b_2, {}^{n}3b_3)$ burst correcting code. The proof is based on the fact that the number of cosets is at least as large as the number of error patterns to be corrected. An $(n=n_1+n_2+n_3, k)$ code is taken as a sub-space of all n-tuples over GF(q) and the weight of a vector and the distance between two vectors shall be considered in the Hamming way.

Theorem 1: The number of parity-check digits required for an $(n=n_1+n_2+n_3,k)$ linear $({}^{n}1b_1, {}^{n}2b_2, {}^{n}3b_3)$ code that correct all bursts of length b_1 (fixed)in first n_1 digits , all bursts of length b_2 (fixed) in next n_2 digits and all bursts of length b_3 (fixed) in the last n_3 digits is atleast

$$\log_q [1+(n_1-b_1+1)(q-1)q^{b_1-1} + (n_2-b_2+1)(q-1)q^{b_2-1} + (n_3-b_3+1)(q-1)q^{b_3-1}] \quad (1)$$

Proof : The theorem is proved by enumerating all error patterns of length b_1 (fixed) in the first sub-block of length n_1, all error patterns of length b_2 (fixed) in the second sub-block of length n_2 and all error patterns of length b_3 (fixed) in the third sub-block of length n_3 and then comparing their sum with the total number of available cosets q^{n-k} . The total number of error patterns, including the pattern of all zeros, is

$$1+(n_1-b_1+1)(q-1)q^{b_1-1} + (n_2-b_2+1)(q-1)q^{b_2-1} + (n_3-b_3+1)(q-1)q^{b_3-1} \quad (2)$$

For correction, all these error patterns must belong to different cosets. Since there must be at least this number of cosets, the result follows. In other words

$({}^{n}1b_1, {}^{n}2b_2, {}^{n}3b_3)$ Linear Codes over GF(2)

$q^{n-k} \geq$

$1+(n_1-b_1+1)(q-1)q^{b_1-1} + (n_2-b_2+1)(q-1)q^{b_2-1} + (n_3-b_3+1)(q-1)q^{b_3-1}$

Incidentally, it can be shown that the result applies to nonlinear codes also.

Discussion

The discussion is centered around the size of the sub-blocks and length of the bursts. We distinguish between different types of block-wise burst error correcting codes according to the ***sizes of sub-blocks*** and ***lengths of the bursts***, which may occur in different sub blocks. There are three different possibilities for the length of burst errors in three sub-blocks viz.

(a) The length of the burst errors is same in every sub-block

(b) The length of the burst errors is different in every sub-block

(c) The length of the burst errors is same in two sub-blocks but different in the third sub-block.

Similarly, there are three different possibilities for the size of the sub-blocks viz.

(a) All sub-blocks are of same size.

(b) All sub-blocks are of different sizes.

(c) Two sub-blocks are of the same size and the third is different.

In this way, these possibilities give rise to a total of 9 different type of bounds with respect to the theorem 1 that listed below in cases 1 to 9. Two special cases 10 and 11 are also discussed in the end.

Case 1. Taking $b_1 = b_2 = b_3 = b$ and $n_1 = n_2 = n_3 = N$.

In this case, we get the expression in (1) as

$$2^{3N-k} \geq 1 + 2^{b-1}\, 3\,(N-b+1)\}. \quad (3)$$

This gives a bound, when *the size of the sub-blocks is samei.e.n/3* and the bursts to be corrected are also having the same length.

Case 2. Taking $b_1 = b_2 = b_3 = b$ and $n_1 \neq n_2 \neq n_3$.

In this case, we get the expression in (1) as

$$2^{n-k} \geq 1 + 2^{b-1}\,(n - 3b + 3). \quad (4)$$

This gives a bound when *the sizes of the sub- blocks are different* whereas the bursts to be corrected are of the same length.

Case 3. Taking $b_1 = b_2 = b_3 = b$ and $n_1 \neq n_2 = n_3$ etc.

In this case, we get expression in (2) as

$$2^{n-k} \geq 1 + 2^{b-1}(n - 3b + 3). \tag{5}$$

This gives a bound when *the size of two sub-blocks are same and the third is different* and the bursts to be corrected are of the same length is each sub-block.

Note: **Cases 1,2,3** discussed above are also referred to as byte-oriented codes [7].

Case 4. Taking $b_1 \neq b_2 \neq b_3$ and $n_1 = n_2 = n_3 = N$
In this case, we get the expression in (1) as

$$2^{3N-k} \geq 1 + (\mathbf{N}-b_1+1)\,2^{b_1-1} + (\mathbf{N}-b_2+1)\,2^{b_2-1} + (\mathbf{N}-b_3+1)\,2^{b_3-1}. \tag{6}$$

This gives a bound when *the size of the sub-blocks are same* and the bursts to be corrected are different in each sub-block.

Case 5. Taking $b_1 \neq b_2 \neq b_3$; $n_1 \neq n_2 \neq n_3$.

In this case, we get the expression in (1) as

$$2^{n-k} \geq 1+(n_1-b_1+1)2^{b_1-1}+(n_2-b_2+1)2^{b_2-1}+(n_3-b_3+1)2^{b_3-1} \tag{7}$$

This gives a bound when both *the size of the sub-blocks* and the length of the bursts to be corrected are different .

Case 6. Taking $b_1 \neq b_2 \neq b_3$; $n_1 = n_2 \neq n_3$ etc.

In this case, there will be three different situations depending upon the length of the blocks and for one such situation, the expression in (1) will look like

$$2^{n-k} \geq 1+(\mathbf{n_1}-b_1+1)2^{b_1-1}+(\mathbf{n_1}-b_2+1)2^{b_2-1}+(n_3-b_3+1)2^{b_3-1} \tag{8}$$

This gives a bound when the size of two sub-blocks is same and the bursts to be corrected are of different lengths.

Case 7. Taking $b_1 = b_2 \neq b_3$; $n_1 = n_2 = n_3 = N$.

In this case, there will be three different situations depending upon the value of the bursts and for one such situation, the expression in (1) will look like

$$2^{3N-k} \geq 1+(N-\mathbf{b_1}+1)2^{b_1-1} + (N-\mathbf{b_1}+1)2^{b_1-1} +(N-b_3+1)2^{b_3-1}. \tag{9}$$

This gives a bound when the size of the sub-blocks is same and the bursts to be corrected are same in two sub-blocks but different in the third.

Case 8. Taking $b_1 = b_2 \neq b_3$, $n_1 \neq n_2 \neq n_3$ etc.

$(^{n}1b_1, {}^{n}2b_2, {}^{n}3b_3)$ Linear Codes over GF(2)

In this case, there will be three different situations depending upon the value of bursts and for one such situation, the expression in (1) will look like

$$2^{n-k} \geq 1 + (n_1 - \mathbf{b_1} + 1)\, 2^{b_1-1} + (n_2 - \mathbf{b_1} + 1)2^{b_1-1} + (n_3 - b_3 + 1)\, 2^{b_3-1} \qquad (10)$$

This gives a bound when *the size of all the sub-blocks are different* and the bursts to be corrected are of the same length in two sub-blocks and different in third.

Case 9. Taking $b_1 = b_2 \neq b_3$, $n_1 = n_2 \neq n_3$.

In this case, there will be three different situations depending upon the value of the bursts and size of the sub-blocks and for one such situation, the expression in (1) will look like

$$2^{n-k} \geq 1 + (\mathbf{n_1} - \mathbf{b_1} + 1)\, 2^{b_1-1} + (\mathbf{n_1} - \mathbf{b_1} + 1)2^{b_1-1} + (n_3 - b_3 + 1)\, 2^{b_3-1} \qquad (11)$$

This gives a bound when the size of only two sub-blocks is same and that the bursts to be corrected are also same in two sub-blocks.

Case 10. When the length of the third block is zero i.e. $n_3 = 0$

Then obviously the expression in (1) will result into

$$q^{n-k} \geq 1+(n_1-b_1+1)\,(q-1)q^{b_1-1} + (n_2-b_2+1)(q-1)q^{b_2-1} \qquad (12)$$

which coincides with a result due to Dass and Tyagi [4].

Case 11. When both $n_2 = n_3 = 0$.

Then, the expression in (2) reduces to

$$q^{n-k} \geq 1+(n-b+1)\,(q-1)q^{b-1}, \qquad (13)$$

which is a bound due to Dass [3].

A Sufficient Condition

We now obtain an upper bound over the number of parity check digits required for the existence of $(^{n}1b_1, {}^{n}2b_2, {}^{n}3b_3)$ - (n, k) linear codes studied in section 1. The proof involves the construction of a matrix that serves the purpose of a parity check matrix for the requisite code and ensures its existence.

Theorem 2. Given positive integers b_1, b_2 and b_3; there exists an $(^{n}1b_1, {}^{n}2b_2, {}^{n}3b_3)$ linear code that correct all bursts of length b_1 (fixed) in the first sub-block of length n_1, all bursts of length b_2(fixed) in the second sub-block of length n_2 and all bursts of length b_3 (fixed) in the third sub-block of length n_3 satisfying the inequality.

$$q^{n-k} \geq \max [q^{b_3-1} \{1+(n_3-2b_3+1)(q-1)q^{b_3-1}\}, q^{b_2-1}\{1+(n_2-2b_2+1)(q-1)q^{b_2-1} + (n_3-b_3+1)(q-1)q^{b_3-1}\}, q^{b_1-1}\{1+(n_1-2b_1+1)(q-1)q^{b_1-1} +(n_2-b_2+1)(q-1)q^{b_2-1} + (n_3-b_3+1)(q-1)q^{b_3-1}\}] \quad (14)$$

Proof. The existence of such a code is shown here by constructing an appropriate (n-k) x n parity check matrix H for the desired code. If H_1' denotes the number of columns of the parity-check matrix H in the first n_1 digits,H_2' denotes the number of columns of H in next n_2 digits and H_3' denotes the number of columns of H in last n_3 digits , then the matrix H may be expressed as $H = [H_1'\ H_2'\ H_3']$. We shall, in fact, construct a matrix H', from which we shall obtain the required matrix H by reversing the order of the columns of H' where $H' = [H_3'\ H_2'\ H_1']$.

Select any non zero (n-k) tuple as the first column of H'(in H_3')..Subsequent columns are added to H' such that after having selected n_3-1 columns h_1, h_2, ---, h_{n_3-1}, a column h_{n_3} is added provided that

hn3≠(un3-b3+1hn3-b3+1+----+un3-1hn3-1)+(vihi+----+vi+b3-1hi+b3-1) (15)

where either all the v_i are zero or if v_s is the last non-zero coefficient, then $b_3 \leq s \leq n_3-b_3$.

This constraint assures that the code which is the null space of the finally constructed matrix H will be capable of correcting all bursts of length b_3 (fixed) in the third sub-block of length n_3. The number of ways in which u_j can be selected is q^{b_3-1}. To choose the v_i is equivalent to enumerating the number of bursts of length b_3 (fixed) in an (n_3-b_3) tuple. Their number (refer to theorem 1) is

$$(n_3-2b_3+1)(q-1)q^{b_3-1} \quad (16)$$

Thus the total number of columns to which h_{n_3} cannot be equal is

$$q^{b_3-1}[1 + (n_3 - 2b_3 + 1)(q-1)q^{b_3-1}] \quad (17)$$

Now we shall start adding $(n_3+1)^{th}$, $(n_3+2)^{th}$,------ columns to H'(in H_2').

We wish to assure that the code so constructed is capable of correcting all bursts of length b_2 (fixed) in the second sub-block of length n_2. For that we lay down following two requirements.

As the first requirement, the general t^{th} column ($t > n_3$) to be added should not be a linear combination of the immediately preceding b_2-1 columns $h_{t-b_2+1}, \ldots, h_{t-1}$ together with any b_2 consecutive columns amongst h_{n_3+1}, h_{n_3+2}, ------, h_{t-b_2} i.e.

$$h_t \neq (u_{t-b_2+1} h_{t-b_2+1} + + u_{t-1} h_{t-1}) + (v_r h_r + + v_{r+b_2-1} h_{r+b_2-1}) \quad (18)$$

where h_r are amongst $h_{n_3+1}, h_{n_3+2}, \ldots, h_{t-b_2}$, and either all the v_r are zero or if v_t is the last non zero coefficient, then $b_2 \leq t \leq t - n_3 - b_2$.

The u_t in (18) can obviously be selected in q^{b_2-1} ways. Choosing the v_t in (18) is equivalent to choosing the number of bursts of length b_2 (fixed) in a vector of length $t - n_3 - b_2$. Their number is

$$1 + (t-n_3 - 2b_2 + 1)(q- 1)\, q^{b_2-1} \quad (19)$$

The second requirement is that the t^{th} column should also not be a linear combination of the immediately preceding b_2-1 columns $h_{t-b_2+1}, \ldots h_{t-1}$ ($t-b_2 + 1 \geq n_3 + 1$) together with any b_3 consecutive columns from amongst $h_1, h_2, \ldots, h_{n_3}$, i.e.

$$h_t \neq (u_{t-b_2+1} h_{t-b_2+1} + \ldots + u_{t-1} h_{t-1}) + (v_i h_i + \ldots + v_{i+b_3-1} h_{i+b_3-1}) \quad (20)$$

where all the v_i are not zero, and if v_s is the last non-zero coefficient, then $b_3 < s$. The number of ways in which the coefficient u_t in (20) can be selected is q^{b_2-1}. Choosing the coefficient v_i in (20) is equivalent to enumerating the bursts of length b_3 (fixed) in a vector of length n_3. Their number is

$$(n_3-b_3+1)\,(q-1)\, q^{b_3-1}. \quad (21)$$

So the total number of combination to which h_t cannot be equal is (19) + (21), i.e.

$$q^{b_2-1} [\, 1+(t-n_3-2b_2+1)\,(q-1)\, q^{b_2-1} + (n_3-b_3+1)\,(q-1)\, q^{b_3-1}] \quad (22)$$

Taking $t = n_3 + n_2$ as the last column of the second sub-block, the equation (22) becomes

$$q^{b_2-1} [\, 1+(n_2-2b_2+1)\,(q-1)\, q^{b_2-1} + (n_3-b_3+1)\,(q-1)\, q^{b_3-1}] \quad (23)$$

The first requirement assures that in the code which is the null space of the finally constructed matrix H, the syndromes of any two bursts each of which is of length b_2 (fixed) are not equal, whereas the second requirement assures that the syndromes of two bursts ,one of which is a burst of length b_2 (fixed) in the sub-block of length n_2 and the other bursts of length b_3 (fixed) in the sub-block of length n_3, are different.

Now, we shall start adding $(n_3 + n_2 + 1)^{th}$, $(n_3 + n_2 + 2)^{th}$, ..., columns to $H'(H_1')$. We wish to assure that the code so constructed is capable of correcting all bursts of length b_1 (fixed) in the first sub-block of length n_1. For this, we lay down the following three requirements.

As the first requirement, the general k^{th} column ($k > n_3 + n_2$) to be added should not be a linear combination of the immediately preceding b_1-1 columns h_{k-b_1+1}, ..., h_{k-1} ,($k-b_1+1 \geq n_3 + n_2+1$) together with any b_1 consecutive columns from amongst

$h_{n_3+n_2+1}$, $h_{n_3+n_2+2}$, ..., h_{k-b_1}, i.e.

$$h_k \neq (u_{k-b_1+1}h_{k-b_1+1}+---+u_{k-1}\, h_{k-1})+(v_r\, h_r+---+v_{r+b_1-1}h_{r+b_1-1}) \qquad (24)$$

where h_r are amongst $h_{n_3+n_2+1}$, $h_{n_3+n_2+2}$, ..., h_{t-b_1}, and either all the v_r are zero or if v_k is the last non-zero coefficient, then

$b_1 \leq k \leq k - (n_3 + n_2) - b_1$.

The u_k in (24) can obviously be selected in q^{b_1-1} ways. Choosing the v_r in (24) is equivalent to choosing the number of bursts of length b_1 (fixed) in a vector of length $k - (n_3 + n_2) - b_1$. Their number is

$$1 + (k - n_3 - n_2 - 2b_1+1)\,(q-1)\,q^{b_1-1} \qquad (25)$$

The second requirement is that the k^{th} column should also not be a linear combination of the immediately preceding b_1-1 columns h_{k-b+1},....., h_{k-1}

($k-b_1+1 \geq n_3 + n_2 + 1$) together with any b_2 consecutive columns from amongst h_{n_3+1}, h_{n_3+2}, ..., $h_{n_3+n_2}$ i.e.

$$h_k \neq (u_{k-b_1+1}h_{k-b_1+1}+....+u_{k-1}h_{k-1})+(v_i h_i+....+v_{i+b_2-1}h_{i+b_2-1}) \qquad (26)$$

where all the v_i are not zero, and if v_s is the last non zero coefficient, then $b_2 \leq s$. The number of ways in which the coefficient u_k in (26) can be selected is

q^{b_2-1}. Choosing the coefficient v_i in (26) is equivalent to enumerating the bursts of length b_2 (fixed) in a vector of length n_2. Their number is

$$(n_2 - b_2 + 1)\,(q-1)\,q^{b_2-1} \qquad (27)$$

The third requirement is that the k^{th} column should also not be a linear combination of the immediately preceding b_1-1 columns $h_{k-b+1}, \ldots, h_{k-1}$ $(k-b_1+1 \geq n_3 + n_2 + 1)$ together with any b_3 consecutive columns from amongst $h_1, h_2, \ldots, h_{n_3}$, i.e.

$$h_k \neq (u_{k-b_1+1}h_{k-b_1+1} + \ldots + u_{k-1}h_{k-1}) + (v_ih_i + \ldots + v_{i+b_3-1}h_{i+b_3-1}) \qquad (28)$$

where all the v_i are not zero, and if v_s is the last non-zero coefficient, then $b_3 \leq s$. The number of ways in which the coefficient u_k in (28) can be selected is q^{b_3-1}. Choosing the coefficient v_i in (28) is equivalent to enumerating the bursts of length b_3 (fixed) in a vector of length n_3. Their number is

$$(n_3-b_3 + 1)\,(q-1)\,q^{b_3-1} \qquad (29)$$

So the total number of combination to which h_k cannot be equal is (25) + (27) + (29) i.e.

$$q^{b_1-1}[1+(k-n_3-n_2-2b_1+1)(q-1)q^{b_1-1} + (n_2-b_2+1)(q-1)q^{b_2-1} + (n_3-b_3+1)(q-1)q^{b_3-1}] \qquad (30)$$

Taking $k = n_3 + n_2 + n_1$ as the last column of the first sub-block, equation (30) becomes

$$q^{b_1-1}[1+(n_1-2b_1+1)\,(q-1)\,q^{b_1-1} + (n_2-b_2+1)\,(q-1)\,q^{b_2-1} + (n_3-b_3+1)\,(q-1)\,q^{b_3-1}] \qquad (31)$$

The first requirement assures that in the code which is the null space of the finally constructed matrix H, the syndromes of any two bursts each of which is of length b_1 (fixed) are not equal, the second requirement assures that the syndromes of two bursts, one of which is a burst of length b_1 (fixed) in the sub-block of length n_1 and the other is a burst of length b_2 (fixed) in the sub-block of length n_2, are different and the third requirement assures that the syndromes of two bursts, one of which is a burst of length b_1 (fixed) in the sub- block of length n_1 and the other is a burst of length b_3 (fixed) in the third block of length n_3, are different.

At worst of all these linear combination considered in (17), (23) and (31) may be distinct. Thus while choosing the n_3^{th} column, we must have

$$q^{n-k} \geq (17); \quad (32)$$

while choosing the $(n_3 + n_2)^{th}$ column, we must have

$$q^{n-k} \geq (23); \quad (33)$$

whereas while choosing the n^{th} column $(n_3 + n_2 + n_1 = n)$, we must have

$$q^{n-k} \geq (31) \quad (34)$$

However the requisite matrix H' can be completed if

$$q^{n-k} \geq \max \{(32), (33), (34)\};$$

which is (14). The required parity check matrix

$H=[H_1' \; H_2' H_3'] = [h_1, h_2, ..., h_n]$ is then the matrix obtained from

$H' = [H_3' H_2' \; H_1'] = [h_n \, h_{n-1} \ldots\ldots h_2 \, h_1]$by reversing its columns altogether i.e. h_j becomes h_{n-j+1}

Discussion

As discussed in the previous section ,here also we get 11 different possibilities for the upper bound given in expression (14). They are listed below in cases 1 to 11 .

Case 1. Taking $b_1 = b_2 = b_3 =b$ and $n_1 = n_2 = n_3= N$, we get the expression in (14) as

$$q^{n-k} \geq \max [q^{b-1} \{1+(N-2b+1) (q-1) q^{b-1}\}; \; q^{b-1} \{1+(2N-3b+2) (q-1) q^{b-1}\}; \; q^{b-1} \{1+(3N-4b+3) (q-1) q^{b-1}\}] \quad (35)$$

This gives an upper bound, when the size of the blocks is same and the bursts to be corrected are also having the same length.

Case 2. Taking $b_1 = b_2 = b_3 = b$ and $n_1 \neq n_2 \neq n_3$, the expression in (14) becomes

$$q^{n-k} \geq \max [q^{b-1} \{1+(n_3-2b+1) (q-1) q^{b-1}\}; \; q^{b-1} \{1+(n-n_1-3b+2) (q-1)q^{b-1}\}; \quad (36)$$

$q^{b-1}\{1+(n-4b+3)(q-1)q^{b-1}\}]$

This gives a bound when the size of the sub-blocks is different where as the bursts to be corrected are of the same length.

Case 3. Taking $b_1 = b_2 = b_3 = b$ and $n_1 = n_2 \neq n_3$, the expression (14) becomes

$$q^{n-k} \geq \max \Big[q^{b-1}\{1+(n_3-2b+1)(q-1)q^{b-1}\};\; q^{b-1}\{1+(n-n_1-3b+2)(q-1)q^{b-1}\};\; q^{b-1}\{1+(n-4b+3)(q-1)q^{b-1}\}\Big] \quad (37)$$

This gives a bound when the sizes of the two sub-blocks are same and the third is different and the bursts to be corrected are of the same length in each sub-block

Note: The codes derived from the expression (35),(36) and (37) turns out to be byte correcting codes [7].

Case 4. Taking $b_1 \neq b_2 \neq b_3$ and $n_1 = n_2 = n_3 = N$ (say) we get the expression in (14) as

$$q^{n-k} \geq \max \Big[q^{b_3-1}\{1+(N-2b_3-+1)(q-1)q^{b_3-1}\};\; q^{b_2-1}\{1+(N-2b_2+2)(q-1)q^{b_2-1}+(N-b_3+1)(q-1)q^{b_3-1}\};\; q^{b_1-1}\{1+(N-2b_1+1)(q-1)q^{b_1-1}+(N-b_2+1)(q-1)q^{b_2-1}+(N-b_3+1)(q-1)q^{b_3-1}\}\Big] \quad (38)$$

This gives a bound when the size of the sub-blocks are same and the bursts to be corrected are different.

Case 5. For $b_1 \neq b_2 \neq b_3$ and $n_1 \neq n_2 \neq n_3$ the expression is given in equation (14).

Case 6. For $b_1 \neq b_2 \neq b_3$ and $n_1 = n_2 \neq n_3$, we get the expression in (14) as

$$q^{n-k} \geq \max \Big[q^{b_3-1}\{1+(n_3-2b_3+1)(q-1)q^{b_3-1}\};\; q^{b_2-1}\{1+(n_1-2b_2+1)(q-1)q^{b_2-1}+(n_3-b_3+1)(q-1)q^{b_3-1}\}; \quad (39)$$

$$q^{b_1-1}\{1+(n_1-2b_1+1)(q-1)q^{b_1-1}+(n_1-b_2+1)(q-1)q^{b_2-1}+(n_3-b_3+1)(q-1)q^{b_3-1}\}]$$

This gives a bound when the size of two sub-blocks is same and the bursts to be corrected are of the different lengths.

Case 7. For $b_1 = b_2 \neq b_3$ and $n_1 = n_2 = n_3 = N$(say), then the expression in (14) becomes

$$q^{n-k} \geq \max$$

$$[q^{b_3-1}\{1+(N-2b_3+1)(q-1)q^{b_3-1}\};$$

$$q^{b_1-1}\{1+(N-2b_1+1)(q-1)q^{b_1-1}+(n_3-b_3+1)(q-1)q^{b_3-1}\}; \qquad (40)$$

$$q^{b_1-1}\{1+(2N-3b_1+2)(q-1)q^{b_1-1}+(N-b_3+1)(q-1)q^{b_3-1}\}]$$

This gives a bound when the size of the sub-blocks is same and the bursts to be corrected are of the same length in two sub-blocks only.

Case 8. For $b_1 = b_2 \neq b_3$ and $n_1 \neq n_2 \neq n_3$, we get the expression in (14) as

$$q^{n-k} \geq \max$$

$$[q^{b_3-1}\{1+(n_3-2b_3+1)(q-1)q^{b_3-1}\};$$

$$q^{b_1-1}\{1+(n_2-2b_1+1)(q-1)q^{b_1-1}+(n_3-b_3+1)(q-1)q^{b_3-1}\}; \qquad (41)$$

$$q^{b_1-1}\{1+(n-n_3-3b_1+2)(q-1)q^{b_1-1}+(n_3-b_3+1)(q-1)q^{b_3-1}\}]$$

This gives a bound when the length of the sub-blocks is different and the bursts to be corrected are same in two sub-blocks but different in the third.

Case 9. For $b_1 = b_2 \neq b_3$ and $n_1 = n_2 \neq n_3$, we get the expression in (14) as

$$q^{n-k} \geq \max$$

$$[q^{b_3-1}\{1+(n_3-2b_3+1)(q-1)q^{b_3-1}\};$$

$$q^{b_1-1}\{1+(n_1-2b_1+1)(q-1)q^{b_1-1}+(n_3-b_3+1)(q-1)q^{b_3-1}\}; \qquad (42)$$

$$q^{b_1-1}\{1+(2n_1-3b_1+2)(q-1)q^{b_1-1}+(n_3-b_3+1)(q-1)q^{b_3-1}\}]$$

This gives a bound when the length of the sub-blocks is same and the bursts to be corrected are same in two sub-blocks.

Case 10. For $n_3 = 0$, the bound in (14) reduces to

$$q^{n-k} \geq \max \left[q^{b_2-1} \{1+(n_2-2b_2+1)(q-1)\, q^{b_2-1}\};\ q^{b_1-1} \{1+(n_1-2b_1+1)(q-1)q^{b_1-1} + (n_2-b_2+1)(q-1)\, q^{b_2-1}\} \right] \tag{43}$$

which has been proved by Dass and Tyagi [4].

Case 11. Finally for $n_2 = n_3 = 0$, the result in (14) reduces to

$$q^{n-k} \geq q^{b_1-1} \{1+(n_1-2b_1+1)(q-1)\, q^{b_1-1}\} \tag{44}$$

which has been proved by Dass [3].

We conclude by constructing appropriate parity-check matrices in example 1 to 9 for each case discussed above by the synthesis procedure outlined in the proof of theorem 2 and show the existence of such codes.Examples corresponding to case 10 and 11 have already been given in [4] and [3] respectively.

Example1: For Case 1

For (3_2,3_2,3_2) binary code, the matrix

$$H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

may be considered as the parity check matrix . It can be verified from the following error pattern – syndrome table 1 that the code corrects all bursts of length 2 (fixed) in each sub-block of length 3.

Table 1

Error-Pattern	Syndrome
First sub-block	
100 000 000	11
110 000 000	1101
010 000 000	1110
011 000 000	111
Second sub-block	
000 100 000	1
000 110 000	11
000 010 000	10
000 011 000	1011
Third sub-block	
000 000 100	1000

000 000 110	1100
000 000 010	100
000 000 011	110

Example: For Case 5

For $(2_2,3_3,4_4)$ binary code, the matrix

$$H=\begin{pmatrix} 0 & 0 & & 0 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 & 0 & & 0 & 1 & 0 & 0 \\ 0 & 1 & & 0 & 1 & 0 & & 0 & 0 & 1 & 0 \\ 1 & 1 & & 0 & 1 & 1 & & 0 & 0 & 0 & 1 \\ & & & & & & & & & & \end{pmatrix}$$

may be considered as the parity check matrix . It can be verified from the following error pattern – syndrome table 2 that the code corrects all bursts of length 2 (fixed) in first sub-block of length 2 , all bursts of length 3 (fixed) in second sub-block of length 3 and all bursts of length 4 (fixed) in third sub-block of length 4.

Table 2

Error-Pattern	Syndrome
First sub-block	
10 000 0000	0001
11 000 0000	0010
Second sub-block	
00 100 0000	0100
00 110 0000	0111
00 101 0000	0101
00 111 0000	0110
Third sub-block	

Error-Pattern	Syndrome
00 000 1000	1000
00 000 1100	1100
00 000 1010	1010
00 000 1110	1110
00 000 1001	1001
00 000 1011	1011
00 000 1101	1101
00 000 1111	1111

Example: For Case 8

For $(4_3,3_3,5_2)$ binary code, the matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

may be considered as the parity check matrix . It can be verified from the following error pattern – syndrome table 3 that the code corrects all bursts of length 3 (fixed) in first sub-block of length 4 , all bursts of length 3 (fixed) in second sub-block of length 3 and all bursts of length 5 (fixed) in third sub-block of length 2.

Table 3

Error-Pattern	Syndrome
First sub-block	
1000 000 00000	10000
1100 000 00000	11000
1010 000 00000	10100
1110 000 00000	11100
0100 000 00000	01000
0110 000 00000	01100
0101 000 00000	01010
0111 000 00000	01110
Second sub-block	
0000100 00000	00001
0000 110 00000	00011
0000 101 00000	11001
0000 111 00000	11011
Third sub-block	
0000 000 10000	11111
0000 000 11000	11010
0000 000 01000	00101
0000 000 01100	00111
0000 000 00100	00010
0000 000 00110	10001
0000 000 00010	10011
0000 000 00011	01011

For Case 2 ($b_1=b_2=b_3=b$;$n_1 \neq n_2 \neq n_3$)**,** Case 3($b_1=b_2=b_3=b$;$n_1 = n_2 \neq n_3$), Case 4 ($b_1 \neq b_2 \neq b_3$; $n_1 = n_2 = n_3 = N$), Case 6 ($b_1 \neq b_2 \neq b_3$;$n_1 = n_2 \neq n_3$) **,** Case 7($b_1 = b_2 \neq b_3$;$n_1 = n_2 = n_3 = N$), Case 9($b_1 = b_2 \neq b_3$;$n_1 = n_2 \neq n_3$), following matrices H_2 to H_9 viz .

$$H2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H6 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$H9 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

may be considered as parity check matrices, which give rise to (2_2,3_2,4_2), (3_3,3_3,4_3), (4_2,4_3,4_4), (2_2,3_3,4_4), (3_1,3_2,4_3), (4_3,3_3,,5_2), (4_3,4_3,4_2) and (2_2,2_2,3_3) binary codes respectively

Open Problems and Remarks.

In this paper, we have obtained lower and upper bounds on the number of parity-check digits for **($^{n}1b_1$, $^{n}2b_2$, $^{n}3b_3$)** linear codes. Without following any systematic procedure, we have shown the existence of linear codes for different values of the parameters n_1, n_2, n_3, k, b_1, b_2, b_3 by constructing appropriate parity-check matrices following the synthesis procedure outlined in the proof of Theorem 2. Some of these codes have close proximity with byte-correcting codes.

However, the problem needs further investigation to find the possibilities of the existence of ($^{n}1b_1$, $^{n}2b_2$, $^{n}3b_3$)-linear codes in non-binary cases; find the possibility of the existence of ($^{n}1b_1$, $^{n}2b_2$, $^{n}3b_3$) –optimal codes in binary and non-binary cases.

Acknowledgement

The authors are thankful to the referee for his valuable comments and to Prof. B.K. Dass, Department of Mathematics, University of Delhi, for his illuminating discussions on this paper.

REFERENCES

[1] Alexander, M.A. Cryb, R.M. and Nast, D.W. (1960). 'Capabilities of the Telephone network for data transmission', *Bell. System Tech. J.*, 39(3).

[2] Chien, R.T. and Tang, D.T. (1965), 'On definition of a Burst', *IBM J. Res. Development*, 9(4), 292-93.

[3] Dass, B.K. (1980) 'On a burst error correcting code', J. Info. Optimize. Sciences, Vol.1, pp.291-295.

[4] Dass, B.K. and Tyagi, Vinod (1980), 'Bounds on blockwise Burst Error Correcting Linear Codes', *J. Information Sciences*, vol. 20, 157-164.

[5] Dass,B.K., Iembo,Rosana and Jain,Sapna (2006),`(1,2) optimal codes over GF(5),`J.Interdisciplinary mathematics,vol.9,pp 319-326.

[6] Dass,B.K.,Iembo,Rosana and Jain ,Sapna (2006) ,`(1,2) optimal codes over GF(7),`Quality,Reliability and Information Technology,Trends and future Directions , Narosa pub. House ,India.

[7] Etzion,Tuvi,(1998),` Perfect Byte Correcting codes`, IEEE Transactions on Information Theory,vol.44, pp.3140-3146.

[8] Etzion,Tuvi,(2001),`Constructions for Perfect 2-Burst–correcting Codes`, IEEE Transactions on Information Theory,vol.47, no.6,253-255.

[9] Tyagi ,Vinod and Rana, Navneet (2008) ,`(1,2) Optimal codes over GF(3) ,Advances in theoretical and applied Mathematics (to appear).

[10] Tyagi ,Vinod and Rana, Navneet (200) ,`A family of ($b_1$,$b_2$)-Optimal codes over GF(q)`,Global J. Pure and Applied Mathematics,vol.4, No.3,pp. 193-207.