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ON THE B-SCROLLS WITH TIME-LIKE GENERATING VECTOR IN 3-DIMENSIONAL MINKOWSKI SPACE E_1^3

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ABSTRACT

In this paper, as special ruled surfaces, b-scrolls with space-like directrix are introduced in 3-dimensional Minkowski space E_1^3, [1,3]. The generating vector of b-scroll is time-like V^3 binormal vector of space-like directrix curve. The normal vector, the matrix corresponding to the shape operator, the Gaussian and mean curvatures, fundamental forms I. and II., asymptotic lines and curvature lines of b-scrolls together with striction space are studied.

Keywords: B-scroll, time-like; ruled.surfaces; shape operator.

3-BOYUTLU MINKOWSKI UZAYINDA ZAMANSAL ÜRETEÇLİ B-SCROLLER

ÖZET

Bu çalışmada, E_1^3 3-boyutlu Minkowski uzayında space-like dayanak eğrisi boyunca, bu eğrinin time-like V^3 binormal vektörü tarafından üretilen bir regle yüzey olarak b-scroll tanımlandı, [1] ve [3] . Bu yüzeyin S şekil operatörüne karşılık gelen matris, Gauss eğriliği, ortalama eğriliği hesaplandı. I. ve II. temel formlar, asimptotik çizgileri ve eğrilik çizgilerini veren denklemler ifade edildi.

Anahtar Kelimeler : B-scroll; zamansal; regle yüzey; şekil operatorü.

1-.Introduction

Let $\eta(I)$ be a space-like curve with arc length t in 3-dimensional Minkowski space $\mathbf{E}_1^3$. If $\dot{\eta}(t) = V_1$ is a space-like vector then $\eta(I)$ is a space-like curve, [7] , that is

$$\langle V_1, V_1 \rangle = 1.$$

The Frenet vectors of $\eta(I)$ are V_1, V_2, V_3, where normal vector V_2 is space-like and binormal vector V_3 is time-like .Then

$$\langle V_2, V_2 \rangle = 1 \quad \text{and} \quad \langle V_3, V_3 \rangle = -1$$

$$\langle V_1, V_3 \rangle = \langle V_2, V_3 \rangle = \langle V_1, V_2 \rangle = 0$$

hold.

In 3-dimensional Minkowski space $\mathbf{E}_1^3$, lorentz metric is

$$\langle \ldots, \ldots \rangle = (+,+,-).$$

Hence, the cross product of

$$\vec{a} = (a_1, a_2, a_3) \text{ and } \vec{b} = (b_1, b_2, b_3) \text{ is}$$

$$\vec{a} \wedge \vec{b} = (a_3 b_2 - a_2 b_3, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1),$$

or this product is given by

$$\vec{a} \wedge \vec{b} = -\det \begin{bmatrix} V_1 & V_2 & -V_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \tag{1}$$

[9].

Since V_3 is time-like vector, V_1 and V_2 are space-like vectors in 3-dimensional Minkowski space E_1^3 and Frenet Formulas, [2,4-5] , can be given by the following equations:

$$\begin{aligned} \dot{V}_1 &= k_1 V_2 \\ \dot{V}_2 &= -k_1 V_1 + k_2 V_3 \\ \dot{V}_3 &= k_2 V_2 \end{aligned}$$

whose matrix form is

$$\begin{bmatrix}\dot{V}_1\\ \dot{V}_2\\ \dot{V}_3\end{bmatrix}=\begin{bmatrix}0 & k_1 & 0\\ -k_1 & 0 & k_2\\ 0 & k_2 & 0\end{bmatrix}\begin{bmatrix}V_1\\ V_2\\ V_3\end{bmatrix}. \tag{2}$$

2. The B-scrolls with space-like directrix and time-like generating vector in 3-dimensional Minkowski space E_1^3

Let $\eta(I)$ be a space-like curve with arc length t, and let V_1, V_2, V_3 be the Frenet vectors. The parametrization of b-scroll whose directrix is the space-like curve $\eta(I)$ and generating vector is the time-like binormal vector V3 is

$$\varphi(t,u)=\eta(t)+uV_3(t) \tag{3}$$

in 3-dimensional Minkowski space $\mathbf{E}_1^3$. Let M be the surface whose ordered basis vectors φ_t and φ_u at the point $\eta(t)$ are given by

$$\begin{aligned}\varphi_t &= V_1+uk_2V_2\\ \varphi_u &= V_3\end{aligned} \tag{4}$$

Denote the asymptotic bundle by $A(t)=Sp\{V_2,V_3\}$ and the tangent bundle by $T(t)=Sp\{V_1,V_2,V_3\}$. Since

$$\dim A(t)\neq \dim T(t),$$

there is no an edge space but there is a striction (curve) space , [8]. Let p(t) be any curve with equation

$$p(t)=\eta(t)+u(t)V_3(t) \tag{5}$$

on the surface M. Differantiating $p(t)$ with respect to t

we have

$$\begin{aligned}\dot{p}(t) &= \dot{\eta}+\dot{u}(t)V_3(t)+u(t)\dot{V}_3(t)\\ &= V_1+\dot{u}(t)V_3(t)+u(t)k_2V_2(t)\end{aligned}$$

A solution vector u of the equation

$$\left\langle \dot{p}(t), \frac{d}{dt}\left[u(t)V_3(t)\right]\right\rangle = \left\langle V_1 + \dot{u}V_3 + uk_2V_2, \dot{u}V_3 + uk_2V_2\right\rangle = 0 \qquad (6)$$

is a position vector of striction (curve) space , [10] . This equation implies

$$-\dot{u}^2 + (uk_2)^2 = 0$$
$$\dot{u} = \mp uk_2$$
$$\int \frac{\dot{u}}{u} = \mp \int k_2 \qquad (7)$$

Hence, striction curve has the position vectors $u = c_1 e^{\mp \int k_2}$.

The parametrization of the surface M is

$$\varphi(t,u) = \eta(t) + uV_3(t) .$$

Since

$$\overline{\varphi_t} = \frac{V_1 + uk_2V_2}{\left(u^2k_2^2 + 1\right)^{\frac{1}{2}}} \text{ and } \overline{\varphi_u} = \frac{\varphi_u}{\|\varphi_u\|} = \varphi_u = V_3 ,$$

then

$$\left\langle \overline{\varphi_t}, \overline{\varphi_u}\right\rangle = 0.$$

That is, $\overline{\varphi_t}, \overline{\varphi_u}$ are ortonormal tangent vectors and $\chi(M) = Sp\left\{\overline{\varphi_t}, \overline{\varphi_u}\right\}$, so the normal vector of the surface M is

$$N = \frac{\overline{\varphi_t} \wedge \overline{\varphi_u}}{\left\|\overline{\varphi_t} \wedge \overline{\varphi_u}\right\|} \qquad (8)$$

or

$$N = \frac{\overline{\varphi_t} \wedge \overline{\varphi_u}}{\left\| \overline{\varphi_t} \wedge \overline{\varphi_u} \right\|} = -\det \begin{bmatrix} V_1 & V_2 & -V_3 \\ \frac{1}{(1+u^2k_2^2)^{\frac{1}{2}}} & \frac{uk_2}{(1+u^2k_2^2)^{\frac{1}{2}}} & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \left\| \overline{\varphi_t} \wedge \overline{\varphi_u} \right\| = 1$$

Therefore, we obtain

$$N = \frac{-uk_2V_1 + V_2}{\left(1+u^2k_2^2\right)^{\frac{1}{2}}}. \tag{9}$$

Using this normal vector, we can find the matrix S corresponding to the shape operator. On the other hand we know that

$$\begin{aligned} S(\overline{\varphi_t}) &= \lambda_1 \overline{\varphi_t} + \lambda_2 \overline{\varphi_u} &&\Rightarrow \lambda_1 = \left\langle S(\overline{\varphi_t}), \overline{\varphi_t} \right\rangle \\ & &&\Rightarrow \lambda_2 = \left\langle S(\overline{\varphi_t}), \overline{\varphi_u} \right\rangle \\ S(\overline{\varphi_u}) &= \mu_1 \overline{\varphi_t} + \mu_2 \overline{\varphi_u} &&\Rightarrow \mu_1 = \left\langle S(\overline{\varphi_u}), \overline{\varphi_t} \right\rangle \\ & &&\Rightarrow \mu_2 = \left\langle S(\overline{\varphi_u}), \overline{\varphi_u} \right\rangle . \end{aligned} \tag{10}$$

Using the following formula

$$S(\overline{\varphi_t}) = \frac{1}{\left\|\varphi_t\right\|} \frac{\partial N}{\partial t} = \frac{1}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}} \frac{\partial N}{\partial t}$$

we have

$$S(\overline{\varphi_t}) = -\frac{\left(k_1 + u\dot{k}_2 + u^2k_1k_2^2\right)V_1 + \left(uk_1k_2 + u^2k_2\dot{k}_2 + u^3k_1k_2^3\right)V_2 + \left(-k_2 - u^2k_2^3\right)V_3}{(u^2k_2^2+1)^2}$$

The values of λ_1 and λ_2 are found as

$$\lambda_1 = \left\langle S(\overline{\varphi_t}), \overline{\varphi_t} \right\rangle = \left\langle S(\overline{\varphi_t}), \frac{V_1 + uk_2V_2}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}} \right\rangle = \frac{-k_1 - u\dot{k}_2 - u^2k_1k_2^2}{\left(u^2k_2^2+1\right)^{\frac{3}{2}}} \tag{11}$$

and

$$
\begin{aligned}
\lambda_2 &= \left\langle S(\overline{\varphi_t}), \overline{\varphi_u} \right\rangle \\
&= \frac{k_2 + u^2 k_2^3}{\left(u^2 k_2^2 + 1\right)^2} \left\langle V_3, V_3 \right\rangle \\
&= \frac{k_2 \left(u^2 k_2^2 + 1\right)}{\left(u^2 k_2^2 + 1\right)^2} (-1) \\
\lambda_2 &= \frac{-k_2}{u^2 k_2^2 + 1}
\end{aligned}
\tag{12}
$$

respectively. Using the formula

$$S(\overline{\varphi_u}) = S(\varphi_u) = \frac{\partial N}{\partial u}$$

we compute

$$\frac{\partial N}{\partial u} = -\frac{\left(k_2 + u^2 k_2^3 - u^2 k_2^3\right) V_1 + u k_2^2 V_2}{\left(u^2 k_2^2 + 1\right)^{\frac{3}{2}}}$$

$$S(\overline{\varphi_u}) = \frac{-k_2 V_1 - u k_2^2 V_2}{\left(u^2 k_2^2 - 1\right)^{\frac{3}{2}}}.$$

Thus, for

$$
\begin{aligned}
\mu_1 &= \left\langle S(\overline{\varphi_u}), \overline{\varphi_t} \right\rangle \\
\mu_2 &= \left\langle S(\overline{\varphi_u}), \overline{\varphi_u} \right\rangle
\end{aligned}
$$

the value of μ_1 is

$$\mu_1 = \frac{-k_2}{u^2 k_2^2 + 1} = \lambda_2 \tag{13}$$

and the value of μ_2 is

$$\mu_2 = 0. \tag{14}$$

As a result, the matrix corresponding to the shape operator S is

$$S=\begin{bmatrix}\lambda_1 & \lambda_2\\ \mu_1 & \mu_2\end{bmatrix}=\begin{bmatrix}\lambda_1 & \lambda_2\\ \lambda_2 & 0\end{bmatrix}=\begin{bmatrix}-\dfrac{k_1+u\dot{k}_2+u^2k_1k_2^2}{\left(u^2k_2^2+1\right)^{\frac{3}{2}}} & \dfrac{-k_2}{u^2k_2^2+1}\\ \dfrac{-k_2}{u^2k_2^2+1} & 0\end{bmatrix} \tag{15}$$

3. Gaussian curvature: Gaussian curvature of b-scroll whose directrix is space-like curve and generating vector is time-like binormal vector, is denoted by K, which is non positive in 3-dimensional Minkowski space E_1^3

$$K = \varepsilon \det \overline{S} = \det \overline{S}$$

$$= \frac{-k_2^2}{\left(u^2k_2^2+1\right)^2} \tag{16}$$

where

$$\varepsilon_1 = \langle N, N\rangle = 1$$

and N is the space-like normal of b-scroll. This surfaces is not time-like, [6] .

4. Mean curvature: Mean curvature of b-scroll whose directrix is space-like curve and generating vector is time-like binormal vector, is denoted by H in 3-dimensional Minkowski space E_1^3

$$H = \operatorname{tr} S$$

$$= \lambda_1 = \frac{-k_1-u\dot{k}_2-u^2k_1k_2^2}{\left(u^2k_2^2+1\right)^{\frac{3}{2}}} \tag{17}$$

5. I. Fundamental form: Fundamental form I of b-scroll whose directrix is space-like curve and generating vector is time-like binormal vector, is defined by

$$I=\langle d\varphi, d\varphi\rangle \text{and } d\varphi=\varphi_t dt+\varphi_u du,$$

hence ,

$$I=\langle \varphi_t,\varphi_t\rangle dtdt+2\langle \varphi_t,\varphi_u\rangle dtdu+\langle \varphi_u,\varphi_u\rangle dudu \tag{18}$$

which results in

$$I = \left(u^2 k_2^2 + 1\right) dtdt - dudu, \tag{19}$$

Thus, we can write this quadratic form in matrix form as

$$\bar{I} = \begin{bmatrix} u^2 k_2^2 + 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{20}$$

$$\det \bar{I} = -u^2 k_2^2 - 1.$$

6. II. Fundamental form: Fundamental form II of b-scroll whose directrix is space-like curve and generating vector is time-like binormal vector is defined by

$$II = \langle S(d\varphi), d\varphi \rangle \text{ with } d\varphi = \varphi_t dt + \varphi_u du$$

from which

$$\begin{aligned} II &= \langle S(\varphi_t)dt + S(\varphi_u)du, \varphi_t dt + \varphi_u du \rangle \\ &= \langle S(\varphi_t), \varphi_t \rangle dtdt + \langle S(\varphi_t), \varphi_u \rangle dtdu \\ &\quad + \langle S(\varphi_u), \varphi_t \rangle dudt + \langle S(\varphi_u), \varphi_u \rangle dudu \end{aligned} \tag{21}$$

is computed and hence,

$$II = \lambda_1 \left(u^2 k_2^2 + 1\right)^{\frac{1}{2}} dtdt + 2\lambda_2 \left(u^2 k_2^2 + 1\right)^{\frac{1}{2}} dtdu + 0dudu$$

results in

$$II = -\frac{k_1 + u\dot{k}_2 + u^2 k_1 k_2^2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} dtdt - \frac{2k_2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} dtdu. \tag{22}$$

We can write this quadratic form as a matrix;

$$\overline{II} = \begin{bmatrix} -\dfrac{k_1 + u\dot{k}_2 + u^2 k_1 k_2^2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} & \dfrac{k_2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} \\ \dfrac{k_2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} & 0 \end{bmatrix} \tag{23}$$

and

$$\det \overline{II} = \frac{-k_2^2}{u^2 k_2^2 + 1}.$$

7. Asymptotic lines: In 3-dimensional Minkowski space E_1^3, asymtotic lines of b-scroll whose directrix is space-like curve and generating vector is time-like binormal vector, are the curves that satisfy the following equation:

$$II = \langle S(d\varphi), d\varphi \rangle = 0 \tag{24}$$

or

$$II = -\frac{k_1 + u\dot{k}_2 + u^2 k_1 k_2^2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} dtdt - \frac{2k_2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} dtdu = 0$$

It can be shown that this equation takes the form

$$-\left(\frac{k_1 + u\dot{k}_2 + u^2 k_1 k_2^2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} dt + \frac{2k_2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} du \right) dt = 0$$

after some computations. Therefore, we may investigate the following two cases:

i) If $dt = 0$, then asymtotic lines are the mainlines with equation $t = c_1$.

ii) otherwise the asymtotic lines have the following equation:

$$\frac{k_1 + u\dot{k}_2 + u^2 k_1 k_2^2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} dt + \frac{2k_2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} du = 0$$

8. Curvature lines: If T is a tangent vector to any curve on the surface M in 3-dimensional Minkowski space E_1^3, we know that $T \in \chi(M) = Sp\{\varphi_t, \varphi_u\}$ and the differential equation of the curvature lines satisfy $\overline{S}T = \lambda T$. First the tangent vector T, from which

$$\begin{aligned} d\varphi &= \overline{\varphi}_t dt + \overline{\varphi}_u du \\ &= \left(V_1 + uk_2 V_2\right) dt + V_3 du \end{aligned}$$

$$T = \frac{V_1 + uk_2 V_2}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} dt + V_3 du$$

is computed and hence,

$$T = \frac{dt}{\left(u^2 k_2^2 + 1\right)^{\frac{1}{2}}} \varphi_t + du\,\varphi_u$$

The matrix form of tangent vector T is

$$T = \begin{bmatrix} \dfrac{dt}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}} \\ du \end{bmatrix}.$$

Replacing this into the equation $\overline{S}T = \lambda T$,we get

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{dt}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}} \\ du \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda dt}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}} \\ \lambda du \end{bmatrix}. \tag{28}$$

It can be shown that this equation becomes

$$\left.\begin{aligned} \frac{\lambda_1-\lambda}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}}dt + \lambda_2 du = 0 \\ \frac{\lambda_2}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}}dt - \lambda du = 0 \end{aligned}\right\}$$

after some computations.Therefore we have the following equations

$$\left.\begin{aligned} \lambda(\lambda_1-\lambda)\frac{dt}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}} + \lambda_2^2\frac{dt}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}} = 0 \\ \left[\lambda(\lambda_1-\lambda)+\lambda_2^2\right]\frac{dt}{\left(u^2k_2^2+1\right)^{\frac{1}{2}}} = 0 \end{aligned}\right\} \tag{29}$$

where there are two cases:

i) If $dt = 0$, curvature lines are the mainlines with equation $\Rightarrow t = c_1$.

ii) Otherwise the quadratic equation $-\lambda^2 + \lambda_1\lambda + \lambda_2^2 = 0$ is always positive since $\Delta_\lambda = \lambda_1^2 + 4\lambda_2^2$ and so we have two distinct roots

$$\lambda = \frac{-\lambda_1 \mp \sqrt{\lambda_1^2 + 4\lambda_2^2}}{2\lambda_1} \tag{30}$$

If we replace λ_1 and λ_2 into the equation, then the other curvature lines will be obtained.

Example 1: In 3-dimensional Minkowski space E_1^3

$$\eta(t) = (\sqrt{2}\cos t, \sqrt{2}\sin t, t)$$

is a space-like curve. The parametrization of b-scroll with directrix $\eta(t)$ and generating $V_3(t)$ is

$$\varphi(t,u) = \eta(t) + uV_3(t)$$

$$\varphi(t,u) = (\sqrt{2}\cos t - u\sin t, \sqrt{2}\sin t + u\cos t, t + u\sqrt{2})$$

Here, $V_3(t)$ is the time-like binormal vector of the space-like curve $\eta(t)$.

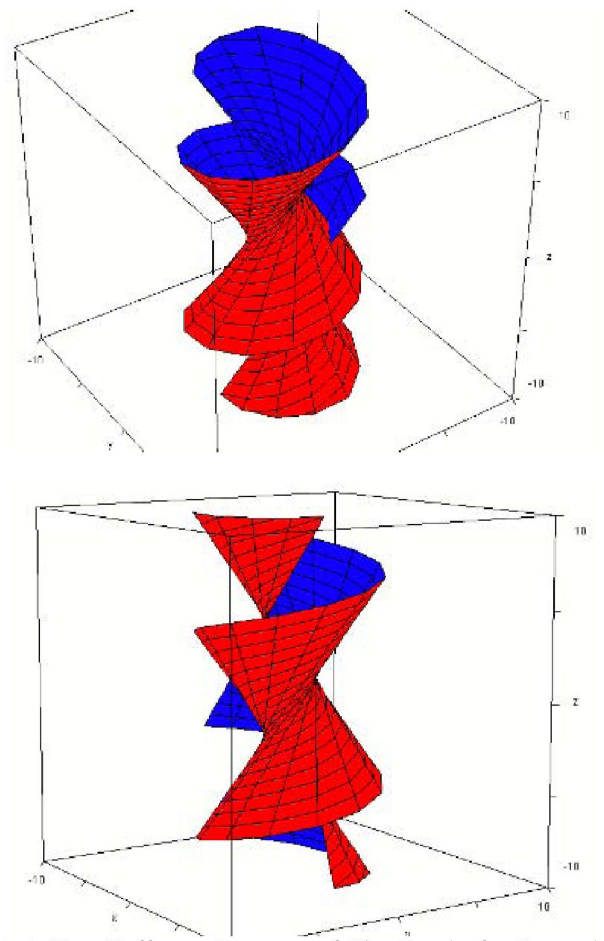


Figure 1, 2: Two Different Positions of The Graphs for Example 1, [11].

Example 2: In 3-dimensional Minkowski space E_1^3

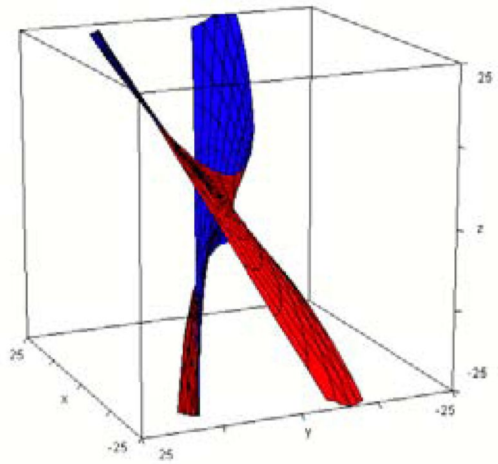
$$\eta(t) = (t - \frac{t^3}{6}, \frac{t^2}{2}, -\frac{t^3}{6})$$

is a space-like curve.The parametrization of b-scroll with directrix $\eta(t)$ and generating $V_3(t)$ is

$$\varphi(t,u) = \eta(t) + uV_3(t)$$

$$\varphi(t,u) = \left(t - \frac{t^3}{6} + u\frac{t^2}{2}, \frac{t^2}{2} - ut, -\frac{t^3}{6} + u + u\frac{t^2}{2}\right)$$

Here, $V_3(t)$ is the time-like binormal vector of the space-like curve $\eta(t)$..



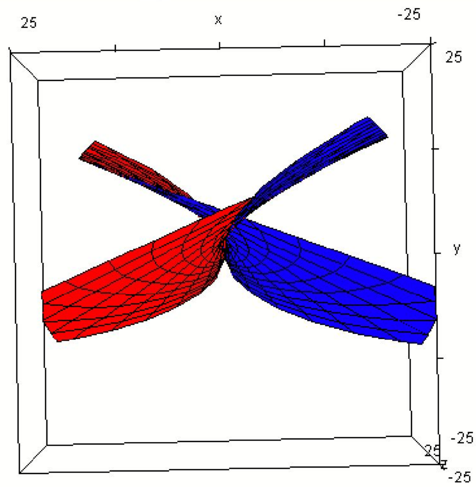


Figure 3, 4: Two Different Positions for Example 2, [11].

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