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On Third-Order Bronze Fibonacci Numbers

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Abstract: In this study, we firstly obtain De Moivre-type identities for the second-order Bronze Fibonacci sequences. Next, we construct and define the third-order Bronze Fibonacci, third-order Bronze Lucas and modified third-order Bronze Fibonacci sequences. Then, we define the generalized third-order Bronze Fibonacci sequence and calculate the De Moivre-type identities for these sequences. Moreover, we find the generating functions, Binet's formulas, Cassini's identities and matrix representations of these sequences and examine some interesting identities related to the third-order Bronze Fibonacci sequences. Finally, we present an encryption and decryption application that uses our obtained results and we present an illustrative example.

Keywords: De Moivre-type identity; third-order Bronze Fibonacci numbers; Binet's formula; Affine-Hill chipher

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1. Introduction

In the literature, the roots of the equation $x^2 - x - 1 = 0$ are given as

$$\alpha_1 = (1+\sqrt{5})/2,$$

$$\alpha_2 = (1-\sqrt{5})/2,$$

and the following relation is satisfied

$$(\frac{1 \pm \sqrt{5}}{2})^n = \frac{L_n \pm \sqrt{5} F_n}{2}, \tag{1}$$

where L_n denotes the n-th Lucas number and F_n denotes the n-th Fibonacci number. Relation (1) is the *De Moivre-type identity* for Fibonacci numbers [1]. Lin, in [2,3], gave the De Moivre-type identities for the tribonacci and the tetranacci numbers by using the equation $x^3 - x^2 - x - 1 = 0$ and the equation $x^4 - x^3 - x^2 - x - 1 = 0$, respectively. Moreover, the authors in [4] obtained the De Moivre-type identities for the second- and third-order Pell numbers by using the roots of characteristic equations $x^2 - 2x - 1 = 0$ and $x^3 - 2x^2 - x - 1 = 0$, respectively. They presented a way to construct the second-order Pell and Pell–Lucas numbers and the third-order Pell and Pell–Lucas numbers. Additionally, in [5], the author studied the generalized third-order Pell numbers. In [6], the authors gave the De Moivre-type identities for the second-order and third-order Jacobsthal numbers.

The second-order Bronze Fibonacci sequence or short Bronze Fibonacci sequence is given by the linear recurrence equation $B_{n+1} = 3B_n + B_{n-1}$ with initial conditions $B_0 = 0$ and $B_1 = 1$; it is also called the 3-Fibonacci Sequence and is defined as the sequence A006190 in the OEIS [7]. In [8], Kartal extended the Bronze Fibonacci numbers to the Gaussian Bronze Fibonacci numbers and obtained Binet's formula and generating functions for these numbers. In [9], the author introduced $(l, 1, p+2q, q)$ numbers, $(l, 1, p+2q, q)$ quaternions, $(l, 1, p+2q, q)$ symbol elements. In [10], the authors presented a special class of elements in the algebras obtained by the Cayley Dickson process, called l-elements or $(l, 1, 0, 1)$ numbers. They gave some properties of these sequences.

It is also known that Fibonacci Numbers are used in encryption theory. In [11], a class of square Fibonacci $(p+1) \times (p+1)$ -matrices, which are based on the Fibonacci p numbers $p = 0, 1, 2, 3, ...$, with a determinant equal to ± 1, was considered. The author defined a Fibonacci coding/decoding method from the Fibonacci matrices which leads to a generalization of the Cassini formula. In [12], the authors present a new method of coding/decoding algorithms using Fibonacci Q matrices. In addition to this, the authors of [13] introduced two new coding/decoding algorithms using Fibonacci Q matrices and R matrices. In [12,13], the used methods are based on the blocked message matrices. In [14], the authors present an application in cryptography and applications of some quaternion elements. In [15], the authors presented a public key cryptosystem using an Affine-Hill chipher with a generalized Fibonacci (multinacci) matrix with large power k, denoted by Q_λ^k, as a key.

In this paper, we give the De Moivre-type identities for the second-order Bronze Fibonacci and the third-order Bronze Fibonacci numbers derived from the characteristic equations $x^2 - 3x - 1 = 0$ and $x^3 - 3x^2 - x - 1 = 0$, respectively. Thus, we define the generalized third-order Bronze Fibonacci numbers, third-order Fibonacci numbers, third-order Bronze Lucas numbers and modified third-order Bronze Fibonacci numbers. We present the generating functions, Binet's formulas, Cassini's identity, matrix representation of third-order Bronze Fibonacci sequences and some interesting identities related to these sequences. Finally, we develop an encryption and decryption algorithm using an Affine-Hill chipher with the third-order Bronze Fibonacci matrix as a key. At the end of paper, we give a numerical example of an encryption and decryption algorithm.

2. De Moivre-Type Identity for the Second- and Third-Order Bronze Fibonacci Numbers

In this section, we firstly obtain De Moivre-type identities for the second-order Bronze Fibonacci numbers. Next, we present a method for constructing the third-order Bronze Fibonacci numbers. We define the third-order Fibonacci numbers, third-order Bronze Lucas numbers, modified third-order Bronze Fibonacci numbers and generalized third-order Bronze Fibonacci numbers. We establish De Moivre-type identities for the third-order Bronze Fibonacci numbers.

The roots of the equation $x^2 - 3x - 1 = 0$ are

$$\alpha_{1,2} = \frac{3 \pm \sqrt{13}}{2}. \tag{2}$$

The De Moivre-type identity for the second-order Bronze Fibonacci numbers can be found as:

$$(\frac{3 \pm \sqrt{13}}{2})^n = \frac{B_n^L \pm \sqrt{13} B_n^F}{2}, \tag{3}$$

where B_n^L represents the Bronze Lucas numbers, which form a Bronze Fibonacci sequence with the initial conditions $B_0^L = 2$ and $B_1^L = 3$, and B_n^F represents Bronze Fibonacci numbers with the initial conditions $B_0^F = 0$ and $B_1^F = 1$.

The third-order Bronze Fibonacci numbers are related to the roots of the equation

$$x^3 - 3x^2 - x - 1 = 0. \tag{4}$$

The three roots of this equation are

$$\alpha_1 = 1 + U + V,$$

$$\alpha_2 = 1 - \frac{1}{2}(U + V) + i\frac{\sqrt{3}}{2}(U - V),$$

$$\alpha_3 = 1 - \frac{1}{2}(U + V) - i\frac{\sqrt{3}}{2}(U - V),$$

where $U = \sqrt[3]{2+\sqrt{4-\frac{64}{27}}}$, $V = \sqrt[3]{2-\sqrt{4-\frac{64}{27}}}$, $UV = \frac{4}{3}$, and $U^3+V^3 = 4$. Thus, the powers of the root α_1 can be calculated as follows:

$$\alpha_1^2 = \frac{11}{3} + 2(U+V) + 1(U^2+V^2),$$

$$\alpha_1^3 = \frac{39}{3} + 7(U+V) + 3(U^2+V^2),$$

$$\alpha_1^4 = \frac{131}{3} + 24(U+V) + 10(U^2+V^2),$$

$$\alpha_1^5 = \frac{443}{3} + 81(U+V) + 34(U^2+V^2),$$

$$\alpha_1^6 = \frac{1499}{3} + 274(U+V) + 115(U^2+V^2).$$

The coefficients of the above equations construct three third-order Bronze Fibonacci sequences, which are denoted by $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{L}\}$, $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{M}\}$ and $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{F}\}$, respectively.

1. $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{L}\}$ is a third-order Bronze Lucas sequence with the recurrence relation $\mathfrak{B}_\mathfrak{n}^\mathfrak{L} = 3\mathfrak{B}_{\mathfrak{n}-1}^\mathfrak{L} + \mathfrak{B}_{\mathfrak{n}-2}^\mathfrak{L} + \mathfrak{B}_{\mathfrak{n}-3}^\mathfrak{L}$ for $n \geq 3$ and $\mathfrak{B}_0^\mathfrak{L} = 3, \mathfrak{B}_1^\mathfrak{L} = 3, \mathfrak{B}_2^\mathfrak{L} = 11$.
2. $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{M}\}$ is a modified third-order Bronze Fibonacci sequence with the recurrence relation $\mathfrak{B}_\mathfrak{n}^\mathfrak{M} = 3\mathfrak{B}_{\mathfrak{n}-1}^\mathfrak{M} + \mathfrak{B}_{\mathfrak{n}-2}^\mathfrak{M} + \mathfrak{B}_{\mathfrak{n}-3}^\mathfrak{M}$ for $n \geq 3$ and $\mathfrak{B}_0^\mathfrak{M} = 1, \mathfrak{B}_1^\mathfrak{M} = 2$ and $\mathfrak{B}_2^\mathfrak{M} = 7$. Additionally, this sequence is also called Bisection of Tribonacci Numbers in OEIS with the code A099463, [7] .
3. $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{F}\}$ is a third-order Bronze Fibonacci sequence with the recurrence relation $\mathfrak{B}_\mathfrak{n}^\mathfrak{F} = 3\mathfrak{B}_{\mathfrak{n}-1}^\mathfrak{F} + \mathfrak{B}_{\mathfrak{n}-2}^\mathfrak{F} + \mathfrak{B}_{\mathfrak{n}-3}^\mathfrak{F}$ for $n \geq 3$ and $\mathfrak{B}_0^\mathfrak{F} = 1, \mathfrak{B}_1^\mathfrak{F} = 3$ and $\mathfrak{B}_2^\mathfrak{F} = 10$. The sequence is also a sum of even indexed terms of Tribonacci Numbers in OEIS with the code A113300 in [7].

The first eleven terms of the above sequences are presented in the Table 1.

Table 1. The third-order Bronze Fibonacci numbers.

N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathfrak{B}_\mathfrak{n}^\mathfrak{L}$	3	3	11	39	131	443	1499	5071	17,155	58,035	196,331
$\mathfrak{B}_\mathfrak{n}^\mathfrak{M}$	1	2	7	24	81	274	927	3136	10,609	35,890	121,415
$\mathfrak{B}_\mathfrak{n}^\mathfrak{F}$	1	3	10	34	115	389	1316	4452	15,061	50,951	172,366

Now, by using these three special third-order Bronze Fibonacci sequences we define a generalized third-order Bronze Fibonacci sequence as follows:

The sequence $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{G}\}$ with the recurrence relation $\mathfrak{B}_\mathfrak{n}^\mathfrak{G} = 3\mathfrak{B}_{\mathfrak{n}-1}^\mathfrak{G} + \mathfrak{B}_{\mathfrak{n}-2}^\mathfrak{G} + \mathfrak{B}_{\mathfrak{n}-3}^\mathfrak{G}$ for $n \geq 3$, where $\mathfrak{B}_0^\mathfrak{G}, \mathfrak{B}_1^\mathfrak{G}, \mathfrak{B}_2^\mathfrak{G}$ are any arbitrary numbers not all being zero, is called a generalized third-order Bronze Fibonacci sequence.

By using the sequences $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{L}\}$, $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{M}\}$, and $\{\mathfrak{B}_\mathfrak{n}^\mathfrak{F}\}$, and applying induction over n, we find

$$\alpha_1^n = \frac{1}{3}\mathfrak{B}_\mathfrak{n}^\mathfrak{L} + \mathfrak{B}_{\mathfrak{n}-1}^\mathfrak{M}(U+V) + \mathfrak{B}_{\mathfrak{n}-2}^\mathfrak{F}(U^2+V^2). \tag{5}$$

Similarly, we obtain

$$\begin{aligned}\alpha_2^n = &\frac{1}{3}\mathfrak{B}_\mathfrak{n}^\mathfrak{L} - \frac{1}{2}\mathfrak{B}_{\mathfrak{n}-1}^\mathfrak{M}(U+V) - \frac{1}{2}\mathfrak{B}_{\mathfrak{n}-2}^\mathfrak{F}(U^2+V^2)\\ &+ \frac{\sqrt{3}i}{2}\mathfrak{B}_{\mathfrak{n}-1}^\mathfrak{M}(U-V) + \frac{\sqrt{3}i}{2}\mathfrak{B}_{\mathfrak{n}-2}^\mathfrak{F}(U^2-V^2),\end{aligned} \tag{6}$$

and

$$\alpha_3^n = \frac{1}{3}\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{L}} - \frac{1}{2}\mathfrak{B}_{\mathfrak{n-1}}^{\mathfrak{M}}(U+V) - \frac{1}{2}\mathfrak{B}_{\mathfrak{n-2}}^{\mathfrak{F}}(U^2+V^2)$$
$$- \frac{\sqrt{3}i}{2}\mathfrak{B}_{\mathfrak{n-1}}^{\mathfrak{M}}(U-V) - \frac{\sqrt{3}i}{2}\mathfrak{B}_{\mathfrak{n-2}}^{\mathfrak{F}}(U^2-V^2). \quad (7)$$

So, we have α_1^n, α_2^n and α_3^n in terms of $\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{L}}$, $\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{M}}$, and $\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{F}}$. Consequently, Equations (5)–(7) are called De Moivre-type identities for the third-order Bronze Fibonacci numbers.

3. Generating Function and Binet's Formula for the Third-Order Bronze Fibonacci Numbers

In this section, we obtain the generating functions and Binet's formulas for the third-order Bronze Fibonacci sequences.

Theorem 1. *The generating function for the generalized third-order Bronze Fibonacci sequence* $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{G}}\}$ *is given by*

$$\mathfrak{B}^{\mathfrak{G}}(x) = \frac{\mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}} + (\mathfrak{B}_{\mathfrak{1}}^{\mathfrak{G}} - 3\mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}})x - (\mathfrak{B}_{\mathfrak{2}}^{\mathfrak{G}} - 3\mathfrak{B}_{\mathfrak{1}}^{\mathfrak{G}} + \mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}})x^2}{1-3x-x^2-x^3}, \quad (8)$$

where $\mathfrak{B}^{\mathfrak{G}}(x) = \sum_{n=0}^{\infty} \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{G}} x^n$.

Proof. Let $\mathfrak{B}^{\mathfrak{G}}(x) = \sum_{n=0}^{\infty} \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{G}} x^n$. By using the recurrence relation, we find

$$\begin{aligned}
\mathfrak{B}^{\mathfrak{G}}(x) &= \sum_{n=3}^{\infty} \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{G}} x^n + \mathfrak{B}_{\mathfrak{2}}^{\mathfrak{G}} x^2 + \mathfrak{B}_{\mathfrak{1}}^{\mathfrak{G}} x + \mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}} \\
&= 3\sum_{n=3}^{\infty} \mathfrak{B}^{\mathfrak{G}}{}_{n-1} x^n + \sum_{n=3}^{\infty} \mathfrak{B}^{\mathfrak{G}}{}_{n-2} x^n + \sum_{n=3}^{\infty} \mathfrak{B}^{\mathfrak{G}}{}_{n-3} x^n + \mathfrak{B}^{\mathfrak{G}}{}_2 x^2 + \mathfrak{B}^{\mathfrak{G}}{}_1 x + \mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}} \\
&= 3x(\mathfrak{B}^{\mathfrak{G}}(x) - \mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}} - \mathfrak{B}_{\mathfrak{1}}^{\mathfrak{G}} x) + x^2(\mathfrak{B}^{\mathfrak{G}}(x) - \mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}}) + x^3\mathfrak{B}^{\mathfrak{G}}(x) + \mathfrak{B}_{\mathfrak{2}}^{\mathfrak{G}} x^2 + \mathfrak{B}_{\mathfrak{1}}^{\mathfrak{G}} x + \mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}}
\end{aligned} \quad (9)$$

and $\mathfrak{B}^{\mathfrak{G}}(x)(1-3x-x^2-x^3) = \mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}} + (\mathfrak{B}_{\mathfrak{1}}^{\mathfrak{G}} - 3\mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}})x + (\mathfrak{B}_{\mathfrak{2}}^{\mathfrak{G}} - 3\mathfrak{B}_{\mathfrak{1}}^{\mathfrak{G}} - \mathfrak{B}_{\mathfrak{o}}^{\mathfrak{G}})x^2$. □

Corollary 1. *The generating functions for the sequences* $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{L}}\}$, $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{M}}\}$ *and* $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{F}}\}$ *can be calculated as follows*

$$\mathfrak{B}^{\mathfrak{L}}(x) = \frac{-x^2-6x+3}{1-3x-x^2-x^3}, \quad (10)$$

where $\mathfrak{B}^{\mathfrak{L}}(x) = \sum_{n=0}^{\infty} \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{L}} x^n$,

$$\mathfrak{B}^{\mathfrak{M}}(x) = \frac{1-x}{1-3x-x^2-x^3}, \quad (11)$$

where $\mathfrak{B}^{\mathfrak{M}}(x) = \sum_{n=0}^{\infty} \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{M}} x^n$ *and*

$$\mathfrak{B}^{\mathfrak{F}}(x) = \frac{1}{1-3x-x^2-x^3}, \quad (12)$$

where $\mathfrak{B}^{\mathfrak{F}}(x) = \sum_{n=0}^{\infty} \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{F}} x^n$.

Theorem 2. *Binet's formula for the generalized third-order Bronze Fibonacci numbers is given by:*

$$\mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}} = \frac{\mathfrak{B}^{\mathfrak{G}}_{\mathfrak{o}}\alpha_2\alpha_3 - \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{1}}(\alpha_2+\alpha_3) + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{2}}}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}\alpha_1^n + \frac{-\mathfrak{B}^{\mathfrak{G}}_{\mathfrak{o}}\alpha_1\alpha_3 + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{1}}(\alpha_1+\alpha_3) - \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{2}}}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\alpha_2^n + \frac{\mathfrak{B}^{\mathfrak{G}}_{\mathfrak{o}}\alpha_1\alpha_2 - \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{1}}(\alpha_1+\alpha_2) + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{2}}}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\alpha_3^n. \tag{13}$$

Proof. We seek for constants d_1, d_2 and d_3 such that

$$\mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}} = d_1\alpha_1^n + d_2\alpha_2^n + d_3\alpha_3^n.$$

These are found by solving the system of linear equations for $n = 0$, $n = 1$ and $n = 2$

$$\begin{aligned} d_1\alpha_1^0 + d_2\alpha_2^0 + d_3\alpha_3^0 &= \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{o}} \\ d_1\alpha_1^1 + d_2\alpha_2^1 + d_3\alpha_3^1 &= \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{1}} \\ d_1\alpha_1^2 + d_2\alpha_2^2 + d_3\alpha_3^2 &= \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{2}}. \end{aligned}$$

□

Corollary 2. *Binet's formulas for the sequences* $\{\mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}}\}$, $\{\mathfrak{B}^{\mathfrak{M}}_{\mathfrak{n}}\}$, *and* $\{\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}}\}$ *can be calculated as:* $\mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}} = \frac{3\alpha_2\alpha_3 - 3(\alpha_2+\alpha_3)+11}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}\alpha_1^n + \frac{-3\alpha_1\alpha_3 + 3(\alpha_1+\alpha_3)-11}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\alpha_2^n + \frac{3\alpha_1\alpha_2 - 3(\alpha_1+\alpha_2)+11}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\alpha_3^n$ *or after making the necessary arrangements*

$$\mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}} = \alpha_1^n + \alpha_2^n + \alpha_3^n, \tag{14}$$

$$\mathfrak{B}^{\mathfrak{M}}_{\mathfrak{n}} = \frac{\alpha_2\alpha_3 - 2(\alpha_2+\alpha_3)+7}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}\alpha_1^n + \frac{-\alpha_1\alpha_3 + 2(\alpha_1+\alpha_3)-7}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\alpha_2^n + \frac{\alpha_1\alpha_2 - 2(\alpha_1+\alpha_2)+7}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\alpha_3^n, \tag{15}$$

$$\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}} = \frac{\alpha_2\alpha_3 - 3(\alpha_2+\alpha_3)+10}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}\alpha_1^n + \frac{-\alpha_1\alpha_3 + 3(\alpha_1+\alpha_3)-10}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\alpha_2^n + \frac{\alpha_1\alpha_2 - 3(\alpha_1+\alpha_2)+10}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\alpha_3^n. \tag{16}$$

4. Some Properties of $\{\mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}}\}$, $\{\mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}}\}$, $\{\mathfrak{B}^{\mathfrak{M}}_{\mathfrak{n}}\}$ and $\{\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}}\}$

In this section, we give some properties of the third-order Bronze Fibonacci sequences such as some equalities and linear sums.

Using the definitions of three third-order Bronze Fibonacci sequences, the following results can be derived easily:

- $\mathfrak{B}^{\mathfrak{M}}{}_{n+1} = \mathfrak{B}^{\mathfrak{F}}{}_{n+1} - \mathfrak{B}^{\mathfrak{F}}{}_{n}$;
- $\mathfrak{B}^{\mathfrak{M}}{}_{n+3} = 2\mathfrak{B}^{\mathfrak{F}}{}_{n+2} + \mathfrak{B}^{\mathfrak{F}}{}_{n+1} + \mathfrak{B}^{\mathfrak{F}}{}_{n}$;
- $\mathfrak{B}^{\mathfrak{L}}{}_{n+2} = 2\mathfrak{B}^{\mathfrak{M}}{}_{n+2} - \mathfrak{B}^{\mathfrak{M}}{}_{n+1} - \mathfrak{B}^{\mathfrak{M}}{}_{n}$;
- $\mathfrak{B}^{\mathfrak{L}}{}_{n+3} = \mathfrak{B}^{\mathfrak{F}}{}_{n+3} + \mathfrak{B}^{\mathfrak{F}}{}_{n+1} + 2\mathfrak{B}^{\mathfrak{F}}{}_{n}$;
- $\mathfrak{B}^{\mathfrak{F}}{}_{n+3} - \mathfrak{B}^{\mathfrak{F}}{}_{n+1} = \mathfrak{B}^{\mathfrak{M}}{}_{n+3} + \mathfrak{B}^{\mathfrak{M}}{}_{n+2}$;
- $\mathfrak{B}^{\mathfrak{L}}{}_{n+4} = 10\mathfrak{B}^{\mathfrak{L}}{}_{n+2} + 4\mathfrak{B}^{\mathfrak{L}}{}_{n+1} + 4\mathfrak{B}^{\mathfrak{L}}{}_{n}$;
- $33\mathfrak{B}^{\mathfrak{L}}{}_{n+4} = -247\mathfrak{B}^{\mathfrak{L}}{}_{n+3} + 134\mathfrak{B}^{\mathfrak{L}}{}_{n+1} + 106\mathfrak{B}^{\mathfrak{L}}{}_{n}$;
- $\mathfrak{B}^{\mathfrak{L}}{}_{n+4} = 4\mathfrak{B}^{\mathfrak{L}}{}_{n+3} - 2\mathfrak{B}^{\mathfrak{L}}{}_{n+2} - \mathfrak{B}^{\mathfrak{L}}{}_{n}$;
- $\mathfrak{B}^{\mathfrak{M}}{}_{n+4} = 10\mathfrak{B}^{\mathfrak{M}}{}_{n+2} + 4\mathfrak{B}^{\mathfrak{M}}{}_{n+1} + 3\mathfrak{B}^{\mathfrak{M}}{}_{n}$;
- $\mathfrak{B}^{\mathfrak{F}}{}_{n+4} = 3\mathfrak{B}^{\mathfrak{F}}{}_{n+2} + 4\mathfrak{B}^{\mathfrak{F}}{}_{n+1} + 10\mathfrak{B}^{\mathfrak{F}}{}_{n}$;
- $\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{M}}{}_{k} = \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}}$.

Theorem 3. *Linear sums for the generalized third-order Bronze Fibonacci numbers are given as follows:*

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{G}}{}_{n+3} - 2\mathfrak{B}^{\mathfrak{G}}{}_{n+2} - 3\mathfrak{B}^{\mathfrak{G}}{}_{n+1} - \mathfrak{B}^{\mathfrak{G}}{}_{2} + 2\mathfrak{B}^{\mathfrak{G}}{}_{1} + 3\mathfrak{B}^{\mathfrak{G}}{}_{0}), \tag{17}$$

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k+1} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{G}}{}_{2n+4} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2n+3} - \mathfrak{B}^{\mathfrak{G}}{}_{2} + 3\mathfrak{B}^{\mathfrak{G}}{}_{1}), \tag{18}$$

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{G}}{}_{2n+3} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2n+2} - \mathfrak{B}^{\mathfrak{G}}{}_{1} + 3\mathfrak{B}^{\mathfrak{G}}{}_{0}). \tag{19}$$

Proof. From the linear recurrence relation of $\mathfrak{B}^{\mathfrak{G}}{}_{n+3}$, we have:

$$\mathfrak{B}^{\mathfrak{G}}{}_{n} = \mathfrak{B}^{\mathfrak{G}}{}_{n+3} - 3\mathfrak{B}^{\mathfrak{G}}{}_{n+2} - \mathfrak{B}^{\mathfrak{G}}{}_{n+1},$$

or

$$\begin{aligned}
\mathfrak{B}^{\mathfrak{G}}{}_{0} &= \mathfrak{B}^{\mathfrak{G}}{}_{3} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2} - \mathfrak{B}^{\mathfrak{G}}{}_{1},\\
\mathfrak{B}^{\mathfrak{G}}{}_{1} &= \mathfrak{B}^{\mathfrak{G}}{}_{4} - 3\mathfrak{B}^{\mathfrak{G}}{}_{3} - \mathfrak{B}^{\mathfrak{G}}{}_{2},\\
\mathfrak{B}^{\mathfrak{G}}{}_{2} &= \mathfrak{B}^{\mathfrak{G}}{}_{5} - 3\mathfrak{B}^{\mathfrak{G}}{}_{4} - \mathfrak{B}^{\mathfrak{G}}{}_{3},\\
&\ldots\\
\mathfrak{B}^{\mathfrak{G}}{}_{n-2} &= \mathfrak{B}^{\mathfrak{G}}{}_{n+1} - 3\mathfrak{B}^{\mathfrak{G}}{}_{n} - \mathfrak{B}^{\mathfrak{G}}{}_{n-1},\\
\mathfrak{B}^{\mathfrak{G}}{}_{n-1} &= \mathfrak{B}^{\mathfrak{G}}{}_{n+2} - 3\mathfrak{B}^{\mathfrak{G}}{}_{n+1} - \mathfrak{B}^{\mathfrak{G}}{}_{n},\\
\mathfrak{B}^{\mathfrak{G}}{}_{n} &= \mathfrak{B}^{\mathfrak{G}}{}_{n+3} - 3\mathfrak{B}^{\mathfrak{G}}{}_{n+2} - \mathfrak{B}^{\mathfrak{G}}{}_{n+1}.
\end{aligned}$$

Summing the left and the right sides of these equations, we obtain:

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{k} = \sum_{k=3}^{n+3} \mathfrak{B}^{\mathfrak{G}}{}_{k} - 3\sum_{k=2}^{n+2} \mathfrak{B}^{\mathfrak{G}}{}_{k} - \sum_{k=1}^{n+1} \mathfrak{B}^{\mathfrak{G}}{}_{k},$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{k} = {} & (\mathfrak{B}^{\mathfrak{G}}{}_{n+1} + \mathfrak{B}^{\mathfrak{G}}{}_{n+2} + \mathfrak{B}^{\mathfrak{G}}{}_{n+3} + \sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{k} - \mathfrak{B}^{\mathfrak{G}}{}_{2} - \mathfrak{B}^{\mathfrak{G}}{}_{1} - \mathfrak{B}^{\mathfrak{G}}{}_{0}),\\
& - 3(\mathfrak{B}^{\mathfrak{G}}{}_{n+1} + \mathfrak{B}^{\mathfrak{G}}{}_{n+2} + \sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{k} - \mathfrak{B}^{\mathfrak{G}}{}_{1} - \mathfrak{B}^{\mathfrak{G}}{}_{0}),\\
& - (\mathfrak{B}^{\mathfrak{G}}{}_{n+1} + \sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{k} - \mathfrak{B}^{\mathfrak{G}}{}_{0}).
\end{aligned}$$

By solving this equation, we obtain

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{G}}{}_{n+3} - 2\mathfrak{B}^{\mathfrak{G}}{}_{n+2} - 3\mathfrak{B}^{\mathfrak{G}}{}_{n+1} - \mathfrak{B}^{\mathfrak{G}}{}_{2} + 2\mathfrak{B}^{\mathfrak{G}}{}_{1} + 3\mathfrak{B}^{\mathfrak{G}}{}_{0}).$$

In the similar way, by using the linear recurrence equation, we find:

$$\begin{aligned}
\mathfrak{B}^{\mathfrak{G}}{}_{1} &= \mathfrak{B}^{\mathfrak{G}}{}_{4} - 3\mathfrak{B}^{\mathfrak{G}}{}_{3} - \mathfrak{B}^{\mathfrak{G}}{}_{2},\\
\mathfrak{B}^{\mathfrak{G}}{}_{3} &= \mathfrak{B}^{\mathfrak{G}}{}_{6} - 3\mathfrak{B}^{\mathfrak{G}}{}_{5} - \mathfrak{B}^{\mathfrak{G}}{}_{4},\\
\mathfrak{B}^{\mathfrak{G}}{}_{5} &= \mathfrak{B}^{\mathfrak{G}}{}_{8} - 3\mathfrak{B}^{\mathfrak{G}}{}_{7} - \mathfrak{B}^{\mathfrak{G}}{}_{6},\\
&\ldots\\
\mathfrak{B}^{\mathfrak{G}}{}_{2n-1} &= \mathfrak{B}^{\mathfrak{G}}{}_{2n+2} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2n+1} - \mathfrak{B}^{\mathfrak{G}}{}_{2n},
\end{aligned}$$

$$\mathfrak{B}^{\mathfrak{G}}{}_{2n+1} = \mathfrak{B}^{\mathfrak{G}}{}_{2n+4} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2n+3} - \mathfrak{B}^{\mathfrak{G}}{}_{2n+2},$$

and by summing side by side, we obtain

$$\begin{aligned}\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k+1} = {} & (\mathfrak{B}^{\mathfrak{G}}{}_{2n+2} + \mathfrak{B}^{\mathfrak{G}}{}_{2n+4} + \sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k} - \mathfrak{B}^{\mathfrak{G}}{}_{0} - \mathfrak{B}^{\mathfrak{G}}{}_{2}) \\ & - 3(\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k+1} - \mathfrak{B}^{\mathfrak{G}}{}_{2n+3} - \mathfrak{B}^{\mathfrak{G}}{}_{1}) \\ & - (\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k} + \mathfrak{B}^{\mathfrak{G}}{}_{2n+2} - \mathfrak{B}^{\mathfrak{G}}{}_{0})\end{aligned}$$

then, solving this equation we obtain:

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k+1} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{G}}{}_{2n+4} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2n+3} - \mathfrak{B}^{\mathfrak{G}}{}_{2} + 3\mathfrak{B}^{\mathfrak{G}}{}_{1}).$$

Similarly, for even indexes, we have

$$\begin{aligned}\mathfrak{B}^{\mathfrak{G}}{}_{0} &= \mathfrak{B}^{\mathfrak{G}}{}_{3} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2} - \mathfrak{B}^{\mathfrak{G}}{}_{1}, \\ \mathfrak{B}^{\mathfrak{G}}{}_{2} &= \mathfrak{B}^{\mathfrak{G}}{}_{5} - 3\mathfrak{B}^{\mathfrak{G}}{}_{4} - \mathfrak{B}^{\mathfrak{G}}{}_{3}, \\ \mathfrak{B}^{\mathfrak{G}}{}_{4} &= \mathfrak{B}^{\mathfrak{G}}{}_{7} - 3\mathfrak{B}^{\mathfrak{G}}{}_{6} - \mathfrak{B}^{\mathfrak{G}}{}_{5}, \\ & \ldots \\ \mathfrak{B}^{\mathfrak{G}}{}_{2n-2} &= \mathfrak{B}^{\mathfrak{G}}{}_{2n+1} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2n} - \mathfrak{B}^{\mathfrak{G}}{}_{2n-1}, \\ \mathfrak{B}^{\mathfrak{G}}{}_{2n} &= \mathfrak{B}^{\mathfrak{G}}{}_{2n+3} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2n+2} - \mathfrak{B}^{\mathfrak{G}}{}_{2n+1},\end{aligned}$$

and $\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k} = (\mathfrak{B}^{\mathfrak{G}}{}_{2n+3} + \sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k+1} - \mathfrak{B}^{\mathfrak{G}}{}_{1}) - 3(\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k} + \mathfrak{B}^{\mathfrak{G}}{}_{2n+2} - \mathfrak{B}^{\mathfrak{G}}{}_{0}) - \sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k+1}$; then, the result is obtained by solving this equation

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{G}}{}_{2k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{G}}{}_{2n+3} - 3\mathfrak{B}^{\mathfrak{G}}{}_{2n+2} - \mathfrak{B}^{\mathfrak{G}}{}_{1} + 3\mathfrak{B}^{\mathfrak{G}}{}_{0}).$$

□

Corollary 3. *Linear sums for the third-order Bronze Lucas sequence* $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{L}}\}$ *are:*

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{L}}{}_{k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{L}}{}_{n+3} - 2\mathfrak{B}^{\mathfrak{L}}{}_{n+2} - 3\mathfrak{B}^{\mathfrak{L}}{}_{n+1} + 4), \tag{20}$$

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{L}}{}_{2k+1} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{L}}{}_{2n+4} - 3\mathfrak{B}^{\mathfrak{L}}{}_{2n+3} - 2), \tag{21}$$

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{L}}{}_{2k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{L}}{}_{2n+3} - 3\mathfrak{B}^{\mathfrak{L}}{}_{2n+2} + 6). \tag{22}$$

Corollary 4. *Linear sums for the modified third-order Bronze Fibonacci sequence* $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{M}}\}$ *are:*

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{M}}{}_{k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{M}}{}_{n+3} - 2\mathfrak{B}^{\mathfrak{M}}{}_{n+2} - 3\mathfrak{B}^{\mathfrak{M}}{}_{n+1}), \tag{23}$$

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{M}}{}_{2k+1} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{M}}{}_{2n+4} - 3\mathfrak{B}^{\mathfrak{M}}{}_{2n+3} - 1), \tag{24}$$

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{M}}{}_{2k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{M}}{}_{2n+3} - 3\mathfrak{B}^{\mathfrak{M}}{}_{2n+2} + 1). \tag{25}$$

Corollary 5. *Linear sums for the third-order Bronze Fibonacci sequence* $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{F}}\}$ *are:*

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{F}}{}_{k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{F}}{}_{n+3} - 2\mathfrak{B}^{\mathfrak{F}}{}_{n+2} - 3\mathfrak{B}^{\mathfrak{F}}{}_{n+1} - 1), \tag{26}$$

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{F}}{}_{2k+1} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{F}}{}_{2n+4} - 3\mathfrak{B}^{\mathfrak{F}}{}_{2n+3} - 1), \tag{27}$$

$$\sum_{k=0}^{n} \mathfrak{B}^{\mathfrak{F}}{}_{2k} = \frac{1}{4}(\mathfrak{B}^{\mathfrak{F}}{}_{2n+3} - 3\mathfrak{B}^{\mathfrak{F}}{}_{2n+2}). \tag{28}$$

5. Cassini's Identity for the Bronze Fibonacci Numbers

In this section, we obtain the well known Cassini identity, sometimes called Simson's formulas, for the third-order Bronze Fibonacci sequences.

Theorem 4. *Cassini's identity for the generalized third-order Bronze Fibonacci numbers is given by*

$$\begin{vmatrix} \mathfrak{B}_{\mathfrak{n}+4}^{\mathfrak{G}} & \mathfrak{B}_{\mathfrak{n}+3}^{\mathfrak{G}} & \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{G}} \\ \mathfrak{B}_{\mathfrak{n}+3}^{\mathfrak{G}} & \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{G}} & \mathfrak{B}_{\mathfrak{n}+1}^{\mathfrak{G}} \\ \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{G}} & \mathfrak{B}_{\mathfrak{n}+1}^{\mathfrak{G}} & \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{G}} \end{vmatrix} = \begin{vmatrix} \mathfrak{B}_{4}^{\mathfrak{G}} & \mathfrak{B}_{3}^{\mathfrak{G}} & \mathfrak{B}_{2}^{\mathfrak{G}} \\ \mathfrak{B}_{3}^{\mathfrak{G}} & \mathfrak{B}_{2}^{\mathfrak{G}} & \mathfrak{B}_{1}^{\mathfrak{G}} \\ \mathfrak{B}_{2}^{\mathfrak{G}} & \mathfrak{B}_{1}^{\mathfrak{G}} & \mathfrak{B}_{0}^{\mathfrak{G}} \end{vmatrix} \tag{29}$$

Proof. By using the induction method, for $n = 1$

$$\begin{vmatrix} \mathfrak{B}^{\mathfrak{G}}{}_5 & \mathfrak{B}^{\mathfrak{G}}{}_4 & \mathfrak{B}^{\mathfrak{G}}{}_3 \\ \mathfrak{B}^{\mathfrak{G}}{}_4 & \mathfrak{B}^{\mathfrak{G}}{}_3 & \mathfrak{B}^{\mathfrak{G}}{}_2 \\ \mathfrak{B}^{\mathfrak{G}}{}_3 & \mathfrak{B}^{\mathfrak{G}}{}_2 & \mathfrak{B}^{\mathfrak{G}}{}_1 \end{vmatrix} = \begin{vmatrix} 3\mathfrak{B}^{\mathfrak{G}}{}_4 + \mathfrak{B}^{\mathfrak{G}}{}_3 + \mathfrak{B}^{\mathfrak{G}}{}_2 & \mathfrak{B}^{\mathfrak{G}}{}_4 & \mathfrak{B}^{\mathfrak{G}}{}_3 \\ 3\mathfrak{B}^{\mathfrak{G}}{}_3 + \mathfrak{B}^{\mathfrak{G}}{}_2 + \mathfrak{B}^{\mathfrak{G}}{}_1 & \mathfrak{B}^{\mathfrak{G}}{}_3 & \mathfrak{B}^{\mathfrak{G}}{}_2 \\ 3\mathfrak{B}^{\mathfrak{G}}{}_2 + \mathfrak{B}^{\mathfrak{G}}{}_1 + \mathfrak{B}^{\mathfrak{G}}{}_0 & \mathfrak{B}^{\mathfrak{G}}{}_2 & \mathfrak{B}^{\mathfrak{G}}{}_1 \end{vmatrix}$$
$$= \begin{vmatrix} \mathfrak{B}^{\mathfrak{G}}{}_2 & \mathfrak{B}^{\mathfrak{G}}{}_4 & \mathfrak{B}^{\mathfrak{G}}{}_3 \\ \mathfrak{B}^{\mathfrak{G}}{}_1 & \mathfrak{B}^{\mathfrak{G}}{}_3 & \mathfrak{B}^{\mathfrak{G}}{}_2 \\ \mathfrak{B}^{\mathfrak{G}}{}_0 & \mathfrak{B}^{\mathfrak{G}}{}_2 & \mathfrak{B}^{\mathfrak{G}}{}_1 \end{vmatrix} = \begin{vmatrix} \mathfrak{B}_{4}^{\mathfrak{G}} & \mathfrak{B}_{3}^{\mathfrak{G}} & \mathfrak{B}_{2}^{\mathfrak{G}} \\ \mathfrak{B}_{3}^{\mathfrak{G}} & \mathfrak{B}_{2}^{\mathfrak{G}} & \mathfrak{B}_{1}^{\mathfrak{G}} \\ \mathfrak{B}_{2}^{\mathfrak{G}} & \mathfrak{B}_{1}^{\mathfrak{G}} & \mathfrak{B}_{0}^{\mathfrak{G}} \end{vmatrix}$$

Let us assume that this identity is true for $n = k$

$$\begin{vmatrix} \mathfrak{B}^{\mathfrak{G}}{}_{k+4} & \mathfrak{B}^{\mathfrak{G}}{}_{k+3} & \mathfrak{B}^{\mathfrak{G}}{}_{k+2} \\ \mathfrak{B}^{\mathfrak{G}}{}_{k+3} & \mathfrak{B}^{\mathfrak{G}}{}_{k+2} & \mathfrak{B}^{\mathfrak{G}}{}_{k+1} \\ \mathfrak{B}^{\mathfrak{G}}{}_{k+2} & \mathfrak{B}^{\mathfrak{G}}{}_{k+1} & \mathfrak{B}^{\mathfrak{G}}{}_{k} \end{vmatrix} = \begin{vmatrix} \mathfrak{B}_{4}^{\mathfrak{G}} & \mathfrak{B}_{3}^{\mathfrak{G}} & \mathfrak{B}_{2}^{\mathfrak{G}} \\ \mathfrak{B}_{3}^{\mathfrak{G}} & \mathfrak{B}_{2}^{\mathfrak{G}} & \mathfrak{B}_{1}^{\mathfrak{G}} \\ \mathfrak{B}_{2}^{\mathfrak{G}} & \mathfrak{B}_{1}^{\mathfrak{G}} & \mathfrak{B}_{0}^{\mathfrak{G}} \end{vmatrix}$$

then, by using the recurrence relation and properties of determinants, we find that (29) is satisfied for $n = k + 1$. □

From this theorem, we give the following corollary:

Corollary 6. *Cassini's identities for the third-ordered Bronze Fibonacci sequences* $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{L}}\}$, $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{M}}\}$, *and* $\{\mathfrak{B}_{\mathfrak{n}}^{\mathfrak{F}}\}$ *are given by*

$$\begin{vmatrix} \mathfrak{B}_{\mathfrak{n}+4}^{\mathfrak{L}} & \mathfrak{B}_{\mathfrak{n}+3}^{\mathfrak{L}} & \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{L}} \\ \mathfrak{B}_{\mathfrak{n}+3}^{\mathfrak{L}} & \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{L}} & \mathfrak{B}_{\mathfrak{n}+1}^{\mathfrak{L}} \\ \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{L}} & \mathfrak{B}_{\mathfrak{n}+1}^{\mathfrak{L}} & \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{L}} \end{vmatrix} = \begin{vmatrix} 131 & 39 & 11 \\ 39 & 11 & 3 \\ 11 & 3 & 3 \end{vmatrix} = -176, \tag{30}$$

$$\begin{vmatrix} \mathfrak{B}_{\mathfrak{n}+4}^{\mathfrak{M}} & \mathfrak{B}_{\mathfrak{n}+3}^{\mathfrak{M}} & \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{M}} \\ \mathfrak{B}_{\mathfrak{n}+3}^{\mathfrak{M}} & \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{M}} & \mathfrak{B}_{\mathfrak{n}+1}^{\mathfrak{M}} \\ \mathfrak{B}_{\mathfrak{n}+2}^{\mathfrak{M}} & \mathfrak{B}_{\mathfrak{n}+1}^{\mathfrak{M}} & \mathfrak{B}_{\mathfrak{n}}^{\mathfrak{M}} \end{vmatrix} = \begin{vmatrix} 81 & 24 & 7 \\ 24 & 7 & 2 \\ 7 & 2 & 1 \end{vmatrix} = -4, \tag{31}$$

$$\begin{vmatrix} \mathfrak{B}^{\mathfrak{F}}_{n+4} & \mathfrak{B}^{\mathfrak{F}}_{n+3} & \mathfrak{B}^{\mathfrak{F}}_{n+2} \\ \mathfrak{B}^{\mathfrak{F}}_{n+3} & \mathfrak{B}^{\mathfrak{F}}_{n+2} & \mathfrak{B}^{\mathfrak{F}}_{n+1} \\ \mathfrak{B}^{\mathfrak{F}}_{n+2} & \mathfrak{B}^{\mathfrak{F}}_{n+1} & \mathfrak{B}^{\mathfrak{F}}_{n} \end{vmatrix} = \begin{vmatrix} 115 & 34 & 10 \\ 34 & 10 & 3 \\ 10 & 3 & 1 \end{vmatrix} = -1, \quad (32)$$

respectively.

6. Matrix Representation of the Third-Order Bronze Fibonacci Numbers

In this section, we give the matrix representation of the the generalized third-order Bronze Fibonacci sequence. Additionally, we derive some properties of this sequence.

The Matrix representation of the generalized third-order Bronze Fibonacci sequence is given by

$$\begin{bmatrix} \mathfrak{B}^{\mathfrak{G}}_{n+3} \\ \mathfrak{B}^{\mathfrak{G}}_{n+2} \\ \mathfrak{B}^{\mathfrak{G}}_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{B}^{\mathfrak{G}}_{n+2} \\ \mathfrak{B}^{\mathfrak{G}}_{n+1} \\ \mathfrak{B}^{\mathfrak{G}}_{n} \end{bmatrix}. \quad (33)$$

By induction over n, we find

$$\begin{bmatrix} \mathfrak{B}^{\mathfrak{G}}_{n+2} \\ \mathfrak{B}^{\mathfrak{G}}_{n+1} \\ \mathfrak{B}^{\mathfrak{G}}_{n} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathfrak{B}^{\mathfrak{G}}_{2} \\ \mathfrak{B}^{\mathfrak{G}}_{1} \\ \mathfrak{B}^{\mathfrak{G}}_{0} \end{bmatrix}.$$

Now, let us define a matrix $\mathbf{B}$ by

$$\mathbf{B} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad (34)$$

Theorem 5. *For $n \geq 4$,*

$$\mathbf{B}^n = \begin{bmatrix} \mathfrak{B}^{\mathfrak{F}}_{n} & \mathfrak{B}^{\mathfrak{F}}_{n-1} + \mathfrak{B}^{\mathfrak{F}}_{n-2} & \mathfrak{B}^{\mathfrak{F}}_{n-1} \\ \mathfrak{B}^{\mathfrak{F}}_{n-1} & \mathfrak{B}^{\mathfrak{F}}_{n-2} + \mathfrak{B}^{\mathfrak{F}}_{n-3} & \mathfrak{B}^{\mathfrak{F}}_{n-2} \\ \mathfrak{B}^{\mathfrak{F}}_{n-2} & \mathfrak{B}^{\mathfrak{F}}_{n-3} + \mathfrak{B}^{\mathfrak{F}}_{n-4} & \mathfrak{B}^{\mathfrak{F}}_{n-3} \end{bmatrix} \quad (35)$$

and $\det \mathbf{B}^n = 1$.

Proof. For $n = 4$, we have

$$\mathbf{B}^4 = \begin{bmatrix} 115 & 44 & 34 \\ 34 & 13 & 10 \\ 10 & 4 & 3 \end{bmatrix} = \begin{bmatrix} \mathfrak{B}^{\mathfrak{F}}{}_4 & \mathfrak{B}^{\mathfrak{F}}{}_3 + \mathfrak{B}^{\mathfrak{F}}{}_2 & \mathfrak{B}^{\mathfrak{F}}{}_3 \\ \mathfrak{B}^{\mathfrak{F}}{}_3 & \mathfrak{B}^{\mathfrak{F}}{}_2 + \mathfrak{B}^{\mathfrak{F}}{}_1 & \mathfrak{B}^{\mathfrak{F}}{}_2 \\ \mathfrak{B}^{\mathfrak{F}}{}_2 & \mathfrak{B}^{\mathfrak{F}}{}_1 + \mathfrak{B}^{\mathfrak{F}}{}_0 & \mathfrak{B}^{\mathfrak{F}}{}_1 \end{bmatrix}.$$

Suppose that for $n = k$

$$\mathbf{B}^k = \begin{bmatrix} \mathfrak{B}^{\mathfrak{F}}{}_k & \mathfrak{B}^{\mathfrak{F}}{}_{k-1} + \mathfrak{B}^{\mathfrak{F}}{}_{k-2} & \mathfrak{B}^{\mathfrak{F}}{}_{k-1} \\ \mathfrak{B}^{\mathfrak{F}}{}_{k-1} & \mathfrak{B}^{\mathfrak{F}}{}_{k-2} + \mathfrak{B}^{\mathfrak{F}}{}_{k-3} & \mathfrak{B}^{\mathfrak{F}}{}_{k-2} \\ \mathfrak{B}^{\mathfrak{F}}{}_{k-2} & \mathfrak{B}^{\mathfrak{F}}{}_{k-3} + \mathfrak{B}^{\mathfrak{F}}{}_{k-4} & \mathfrak{B}^{\mathfrak{F}}{}_{k-3} \end{bmatrix}$$

then,

$$\mathbf{B}^{k+1} = \mathbf{B}^k\mathbf{B} = \begin{bmatrix} \mathfrak{B}^{\mathfrak{F}}{}_k & \mathfrak{B}^{\mathfrak{F}}{}_{k-1} + \mathfrak{B}^{\mathfrak{F}}{}_{k-2} & \mathfrak{B}^{\mathfrak{F}}{}_{k-1} \\ \mathfrak{B}^{\mathfrak{F}}{}_{k-1} & \mathfrak{B}^{\mathfrak{F}}{}_{k-2} + \mathfrak{B}^{\mathfrak{F}}{}_{k-3} & \mathfrak{B}^{\mathfrak{F}}{}_{k-2} \\ \mathfrak{B}^{\mathfrak{F}}{}_{k-2} & \mathfrak{B}^{\mathfrak{F}}{}_{k-3} + \mathfrak{B}^{\mathfrak{F}}{}_{k-4} & \mathfrak{B}^{\mathfrak{F}}{}_{k-3} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$\mathbf{B}^{k+1} = \begin{bmatrix} \mathfrak{B}^{\mathfrak{F}}{}_{k+1} & \mathfrak{B}^{\mathfrak{F}}{}_{k} + \mathfrak{B}^{\mathfrak{F}}{}_{k-1} & \mathfrak{B}^{\mathfrak{F}}{}_{k} \\ \mathfrak{B}^{\mathfrak{F}}{}_{k} & \mathfrak{B}^{\mathfrak{F}}{}_{k-1} + \mathfrak{B}^{\mathfrak{F}}{}_{k-2} & \mathfrak{B}^{\mathfrak{F}}{}_{k-1} \\ \mathfrak{B}^{\mathfrak{F}}{}_{k-1} & \mathfrak{B}^{\mathfrak{F}}{}_{k-2} + \mathfrak{B}^{\mathfrak{F}}{}_{k-3} & \mathfrak{B}^{\mathfrak{F}}{}_{k-2} \end{bmatrix}$$

which proves the theorem. Similarly, by using the properties of determinants and induction over n, we find that $\det \mathbf{B}^n = 1$. □

For $n \geq 4$, let us define a matrix

$$Y_n = \begin{bmatrix} \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-1} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-1} \\ \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-1} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-3} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} \\ \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-3} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-4} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-3} \end{bmatrix}. \quad (36)$$

Theorem 6. *For $n, m \geq 4$*

1. $Y_n = \mathbf{B}^{n-4} Y_4$.
2. $Y_4 \mathbf{B}^n = \mathbf{B}^n Y_4$.
3. $Y_{n+m} = Y_n \mathbf{B}^m$.

Proof.

1. Since $\mathbf{B} Y_n = Y_{n+1}$, it can be easily shown by induction that $Y_n = \mathbf{B}^{n-4} Y_4$.
2. Using the definition of Y_n and induction, we find $Y_4 \mathbf{B}^n = \mathbf{B}^n Y_4$.
3. From 1 and 2, it follows that $Y_{n+m} = \mathbf{B}^{n+m-4} Y_4 = \mathbf{B}^{n-4} \mathbf{B}^m Y_4 = \mathbf{B}^{n-4} Y_4 \mathbf{B}^m = Y_n \mathbf{B}^m$.

□

Theorem 7. *For $n, m \geq 4$, we have*

$$\mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}} = \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-1} (\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1} + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-2}) + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1}. \quad (37)$$

Proof. From the above theorem, we have $Y_{n+m} = Y_n \mathbf{B}^m$, or

$$\begin{bmatrix} \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-1} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-2} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-1} \\ \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-1} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-2} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-3} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-2} \\ \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-2} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-3} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-4} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}-3} \end{bmatrix} =$$
$$\begin{bmatrix} \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-1} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-1} \\ \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-1} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-3} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} \\ \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-2} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-3} + \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-4} & \mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}-3} \end{bmatrix} \begin{bmatrix} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}} & \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1} + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-2} & \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1} \\ \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1} & \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-2} + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-3} & \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-2} \\ \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-2} & \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-3} + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-4} & \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-3} \end{bmatrix}. \quad (38)$$

Since the $\mathfrak{B}^{\mathfrak{G}}_{\mathfrak{n}+\mathfrak{m}}$ entry is the product of the first row of the Y_n and the first column of $\mathbf{B}^n$, the result follows. □

Corollary 7. *For the third-ordered Bronze Fibonacci sequences $\{\mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}}\}$, $\{\mathfrak{B}^{\mathfrak{M}}_{\mathfrak{n}}\}$, and $\{\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}}\}$, we have*

$$\mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}+\mathfrak{m}} = \mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}} + \mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}-1} (\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1} + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-2}) + \mathfrak{B}^{\mathfrak{L}}_{\mathfrak{n}-2} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1}, \quad (39)$$

$$\mathfrak{B}^{\mathfrak{M}}_{\mathfrak{n}+\mathfrak{m}} = \mathfrak{B}^{\mathfrak{M}}_{\mathfrak{n}} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}} + \mathfrak{B}^{\mathfrak{M}}_{\mathfrak{n}-1} (\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1} + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-2}) + \mathfrak{B}^{\mathfrak{M}}_{\mathfrak{n}-2} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1}, \quad (40)$$

$$\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}+\mathfrak{m}} = \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}} + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}-1} (\mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1} + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-2}) + \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{n}-2} \mathfrak{B}^{\mathfrak{F}}_{\mathfrak{m}-1}, \quad (41)$$

respectively.

7. Application: Encryption and Decryption via Third-Order Bronze Fibonacci Numbers

In this section, as a useful application of all obtained results, we give a third-order Bronze Fibonacci encryption and decryption algorithm. In this algorithm, we use the

Affine-Hill chipher method for encryption using a third-order Bronze Fibonacci matrix as a key. First of all, let us list the notations which we use in the encryption and decryption algorithms:

- p is the number of the characters which the sender and receiver use. We chose p to be a prime.
- $\phi(p)$ is the image of the number p under the Euler Phi function. It is known that if p is prime then $\phi(p) = p - 1$.
- D is the private key of the receiver.
- P_1 is any primitive root of p .
- $P_2 = P_1^D \mod p$.
- (p, P_1, P_2) is the public key.
- ε is a positive integer which satisfies $1 < \varepsilon < \phi(p)$. The prime p provides a large key space for the selection of ε. This strengthens the security of the system.
- $\lambda = P_2^{\varepsilon} \mod p$.
- $\mathbf{T} = [\mathfrak{B}^{\mathfrak{F}}{}_{\lambda}\ \mathfrak{B}^{\mathfrak{F}}{}_{\lambda+1}\ \mathfrak{B}^{\mathfrak{F}}{}_{\lambda+2}] \mod p$ is the 1×3 shifting vector.
- $\mathbf{M} = [m_1 m_2 ... m_n]$ is the plain text and $\mathbf{E} = [e_1 e_2 ... e_n]$ is the cipher text. Note that if the plain text has not suitable length in Step 6 of encryption algorithm, zero will be added until it can be divided by 3 with no remainder.
- $\mathbf{M}_i$ is ith vector of type 1×3 obtained using the character table after dividing the plain text into 3-length parts.
- $\mathbf{E}_i$ is ith vector of type 1×3 obtained using the character table after dividing the cipher text $\mathbf{E}$ into 3-length parts.

Note that the prime number p and the large value of λ increase the security of three digital signatures $\lambda, k, \mathbf{T}$. This makes it difficult to break the system.

Encryption Algorithm:

- **Step 1:** The sender chooses a secret number ε where $1 < \varepsilon < \phi(p)$.
- **Step 2:** The sender calculates the signature $k = P_1^{\varepsilon} \mod p$.
- **Step 3:** The sender calculates $\lambda = P_2^{\varepsilon} \mod p$.
- **Step 4:** The sender constructs $\mathbf{B}^{\lambda}$.
- **Step 5:** The sender constructs $\mathbf{T}$.
- **Step 6:** The sender calculates $\mathbf{E}_i = \mathbf{M}_i \mathbf{B}^{\lambda} + \mathbf{T} \mod p$ for $1 \leq i \leq \frac{n}{3}$.
- **Step 7:** The sender sends the cipher text $E = [e_1 e_2 ... e_n]$ and the signature k.

Decryption Algorithm:

- **Step 1:** The receiver calculates $\lambda = k^D \mod p$.
- **Step 2:** The receiver calculates $\mathbf{B}^{\lambda}$ and $\mathbf{B}^{-\lambda}$.
- **Step 3:** The receiver calculates the shifting vector $\mathbf{S}$.
- **Step 4:** The receiver calculates $\mathbf{M}_i = (\mathbf{E}_i - \mathbf{T})\mathbf{B}^{-\lambda} \mod p$ for $1 \leq i \leq \frac{n}{3}$.
- **Step 5:** The receiver constructs the plain text $\mathbf{M} = [m_1 m_2 ... m_n]$.

Example 1. *Let $p = 29$ and consider 29—characters with the numerical values 1–26, 27, 28 and 29 are assigned for the alphabets A–Z, ., 0 and blank space, respectively. Consider that the plain text is "STAY AT HOME", private key is $D = 13$ and $P_1 = 11$. Then, we calculate $P_2 = 21$. So, the public key is (29,11,21).*

Encryption Algorithm:

- ***Step 1:*** *We choose $\varepsilon = 17$ where $1 < \varepsilon < 28$.*
- ***Step 2:*** *The signature $k = 3$.*
- ***Step 3:*** *$\lambda = 19$.*
- ***Step 4:*** *Since $\lambda = 19$ then* $\mathbf{B}^{19} = \begin{bmatrix} 12 & 10 & 27 \\ 27 & 18 & 1 \\ 1 & 24 & 17 \end{bmatrix} \mod 29$, *from Equation (35).*
- ***Step 5:*** *The shifting vector* $\mathbf{T} = [\mathfrak{B}^{\mathfrak{F}}_{19}\ \mathfrak{B}^{\mathfrak{F}}_{20}\ \mathfrak{B}^{\mathfrak{F}}_{21}] = [12\ 6\ 28] \mod 29$. *We can also use the Binet Formula (16) here to calculate the shifting vector* $\mathbf{T}$.

- ***Step 6:***

$$\begin{aligned}
\mathbf{E}_1 &= \mathbf{M}_1\mathbf{B}^{\lambda} + \mathbf{T} \mod 29 \\
&= [19 \quad 20 \;\; 1]\begin{bmatrix} 12 & 10 & 27 \\ 27 & 18 & 1 \\ 1 & 24 & 17 \end{bmatrix} + [12\;6\;28] \mod 29 \\
&= [27 \quad 0 \;\; 27] = [.\,.], \\
\mathbf{E}_2 &= \mathbf{M}_2\mathbf{B}^{\lambda} + \mathbf{T} \mod 29 \\
&= [25 \quad 29 \;\; 1]\begin{bmatrix} 12 & 10 & 27 \\ 27 & 18 & 1 \\ 1 & 24 & 17 \end{bmatrix} + [12\;6\;28] \mod 29 \\
&= [23 \quad 19 \;\; 24] = [WSX], \\
\mathbf{E}_3 &= \mathbf{M}_3\mathbf{B}^{\lambda} + \mathbf{T} \mod 29 \\
&= [20 \quad 29 \;\; 8]\begin{bmatrix} 12 & 10 & 27 \\ 27 & 18 & 1 \\ 1 & 24 & 17 \end{bmatrix} + [12\;6\;28] \mod 29 \\
&= [28 \quad 21 \;\; 8] = [0UH], \\
\mathbf{E}_4 &= \mathbf{M}_4\mathbf{B}^{\lambda} + \mathbf{T} \mod 29 \\
&= [15 \quad 13 \;\; 5]\begin{bmatrix} 12 & 10 & 27 \\ 27 & 18 & 1 \\ 1 & 24 & 17 \end{bmatrix} + [12\;6\;28] \mod 29 \\
&= [26 \quad 17 \;\; 9] = [ZQI].
\end{aligned}$$

- ***Step 7:*** *We send the receiving (cipher) text* $\mathbf{E} = [.\;\;.WSX0UHZQI]$ *and the signature* $k = 3$.
  Decryption Algorithm:
- ***Step 1:*** $\lambda = 3^{13} \mod 29 = 19$.
- ***Step 2:*** *Since* $\lambda = 19$ *then* $\mathbf{B}^{19} = \begin{bmatrix} 12 & 28 & 27 \\ 27 & 18 & 1 \\ 1 & 24 & 17 \end{bmatrix} \mod 29$,
  and $\mathbf{B}^{-19} = \begin{bmatrix} 12 & 8 & 18 \\ 20 & 10 & 12 \\ 12 & 11 & 23 \end{bmatrix} \mod 29$,
- ***Step 3:*** *We calculate the shifting vector* $\mathbf{T} = [12 \quad 6 \quad 28]$.

- ***Step 4:*** *We calculate all* $\mathbf{M}_i$, $i = 1, 2, 3, 4$ *as follows:*

$$\begin{aligned}
\mathbf{M}_1 &= (\mathbf{E}_1 - \mathbf{T})\mathbf{B}^{-19} \mod 29 \\
&= ([13 \;\; 21 \;\; 25] - [12 \;\; 6 \;\; 28]) \begin{bmatrix} 12 & 8 & 18 \\ 20 & 10 & 12 \\ 12 & 11 & 23 \end{bmatrix} \mod 29 \\
&= [19 \;\; 20 \;\; 1] = [STA], \\
\mathbf{M}_2 &= (\mathbf{E}_2 - \mathbf{T})\mathbf{B}^{-19} \mod 29 \\
&= ([23 \;\; 19 \;\; 24] - [12 \;\; 6 \;\; 28]) \begin{bmatrix} 12 & 8 & 18 \\ 20 & 10 & 12 \\ 12 & 11 & 23 \end{bmatrix} \mod 29 \\
&= [25 \;\; 29 \;\; 1] = [Y \;\; A], \\
\mathbf{M}_3 &= (\mathbf{E}_3 - \mathbf{T})\mathbf{B}^{-19} \mod 29 \\
&= ([23 \;\; 19 \;\; 24] - [12 \;\; 6 \;\; 28]) \begin{bmatrix} 12 & 8 & 18 \\ 20 & 10 & 12 \\ 12 & 11 & 23 \end{bmatrix} \mod 29 \\
&= [20 \;\; 29 \;\; 8] = [T \;\; H], \\
\mathbf{M}_4 &= (\mathbf{E}_4 - \mathbf{T})\mathbf{B}^{-19} \mod 29 \\
&= ([26 \;\; 17 \;\; 9] - [12 \;\; 6 \;\; 28]) \begin{bmatrix} 12 & 8 & 18 \\ 20 & 10 & 12 \\ 12 & 11 & 23 \end{bmatrix} \mod 29 \\
&= [15 \;\; 13 \;\; 5] = [OME].
\end{aligned}$$

- ***Step 5:*** *The sending (plain) text* $\mathbf{M} = [STAY\ AT\ HOME]$.

8. Conclusions

In this paper, we define some third-order Bronze Fibonacci sequences. Additionally, we present the De Moivre-type identities for the second- and third-order Bronze Fibonacci numbers. In addition to this, we obtain the generating functions, Binet's Formulas, Cassini's identity, and matrix representation of these sequences and some interesting identities related to the third-order Bronze Fibonacci sequences. Finally, we develop a new third-order Bronze Fibonacci encryption and decryption algorithm in encryption theory.

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