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On Some Properties of Systems of Linear Inequalities for Operators

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Abstract

In this paper, the existence of the solution of linear system of (operator) inequalities is considered using the theory of tensor products of linear spaces, linear operators and some properties of tensor determinantal operators.

Keywords: *tensor determinants; systems of equations; multi-parameter spectral theory*

Özet

Operatörlerin Lineer Eşitsizlik Sistemlerinin Bazı Özellikleri

Bu makalede, lineer uzayların, lineer operatörlerin tensör çarpımından ve tensör determinantlarının bazı özelliklerinden yararlanarak operatörlerin lineer eşitsizlik sistemlerinin çözümünün varlığı ele alındı.

Anahtar Kelimeler: *tensör determinantları; denklem sistemleri; çok parametreli spectral teori*

INTRODUCTION

The tensor determinants are important for the solutions of problems related with multi-parameter spectral problems (MPSP). The system of linear equations

$$b_{i1}\lambda_1 + b_{i2}\lambda_2 + \ldots + b_{in}\lambda_n = a_i, \quad i = 1, 2, \ldots, n \tag{1}$$

has a unique solution if $\det(b_{ik}) \neq 0$, where b_{in}, a_i are fixed numbers. In this paper, we try to find some analogue of this fact in the case, when instead of b_{in}, a_i we have operators in finite dimensional spaces, then we show some application to the MPSP.

Let B_{in}, A_i be self-adjoint linear operators (that is the corresponding matrix is equal to its own conjugate transpose if they act in C^n) in finite dimensional inner-product space H_i, $B_{in}, A_i \in L(H_i)$. The spectrum (or eigenvalues) of MPSP is the set of $(\lambda_1, \lambda_2, ..., \lambda_n) \in C^n$ such that

$$(A_i - \lambda_1 B_{i1} - \lambda_2 B_{i2} - ... - \lambda_n B_{in})x_i = 0, \tag{2}$$

for some $0 \neq x_i \in H_i$, $i = 1, 2, ..., n$. The tensor vector $x_1 \otimes ... \otimes x_n \in H_1 \otimes ... \otimes H_n$ is called an eigenfunction of the problem (2). For the definition and properties of tensor products of spaces, operators and for the multiparameter eigenvalue problems, see [1,2].

We denote by

$$\Delta = \begin{vmatrix} B_{11} & ... & B_{1n} \\ ... & ... & ... \\ B_{n1} & ... & B_{nn} \end{vmatrix} = \otimes \det(B_{ik}), \quad i, k = 1, 2, ..., n,$$

for example if n=2, we have $\Delta = B_{11} \otimes B_{22} - B_{12} \otimes B_{21}$.

Main Result

The next theorem is an interesting improvement of the theorem on the solvability of the system of linear equations.

Theorem 1. If $\dim H_i < \infty$, $i = 1, 2, ..., n$, and $\Delta > 0$, then for any set of self-adjoint operators $A_i \in L(H_i)$ and for any choice of $\varepsilon_i = \pm 1$, $i = 1, 2, ..., n$, there exist a unique $(\lambda_1, \lambda_2, ..., \lambda_n) \in R^n$, such that

$$\varepsilon_i (A_i - \lambda_1 B_{i1} - \lambda_2 B_{i2} - ... - \lambda_n B_{in}) \geq 0, \text{ and} \tag{3}$$

$$Ker(A_i - \lambda_1 B_{i1} - \lambda_2 B_{i2} - ... - \lambda_n B_{in}) \neq \{0\}, \; i = 1, 2, ..., n \tag{4}$$

Proof. For simplicity let us consider the case $\varepsilon_i = 1$, $i = 1, 2, ..., n$. For k =1 we have $B \gg 0$ and therefore, for any operator A, there exists a unique λ such that $A-\lambda B \geq 0$ and $Ker(A-\lambda B) \neq \{0\}$. Let the statement of the theorem be true for $k=n-1$. We shall show that it holds for $k=n$.

Since $\Delta \gg 0$, we can replace B_{jk} by their linear combinations such that these new operators satisfy the condition

$$\begin{vmatrix} B_{22} & \dots & B_{2n} \\ \dots & \dots & \dots \\ B_{n2} & \dots & B_{nn} \end{vmatrix} \gg 0. \tag{5}$$

Indeed, for a fixed $0 \neq x_1 \in H_1$ we obtain

$$\begin{vmatrix} (B_{11}x_1, x_1) & \dots & (B_{1n}x_1, x_1) \\ B_{21} & \dots & B_{2n} \\ \dots & \dots & \dots \\ B_{n1} & \dots & B_{nn} \end{vmatrix} \gg 0.$$

At least one of $(B_{1k}x_1, x_1) \neq 0,$ say $(B_{11}x_1, x_1) \neq 0$. To eliminate the elements $(B_{1k}x_1, x_1),\ k \neq 1$, add $-(B_{1k}x_1, x_1)/(B_{11}x_1, x_1)$ times column1 to column k:

$$\begin{vmatrix} (B_{11}x_1, x_1) & 0 & \dots & 0 \\ B_{21} & B_{22} - \dfrac{(B_{12}x_1, x_1)}{(B_{11}x_1, x_1)} B_{21} & \dots & B_{2n} - \dfrac{(B_{1n}x_1, x_1)}{(B_{11}x_1, x_1)} B_{21} \\ & \dots\dots\dots\dots & & \dots\dots\dots\dots\dots\dots \\ B_{n1} & B_{n2} - \dfrac{(B_{12}x_1, x_1)}{(B_{11}x_1, x_1)} B_{n1} & \dots & B_{nn} - \dfrac{(B_{1n}x_1, x_1)}{(B_{11}x_1, x_1)} B_{n1} \end{vmatrix} \gg 0.$$

Thus the tensor determinant

$$\begin{vmatrix} B_{22}-\dfrac{(B_{12}x_1,x_1)}{(B_{11}x_1,x_1)}B_{21} & \dots & B_{2n}-\dfrac{(B_{1n}x_1,x_1)}{(B_{11}x_1,x_1)}B_{21} \\ \dots & \dots & \dots \\ B_{n2}-\dfrac{(B_{12}x_1,x_1)}{(B_{11}x_1,x_1)}B_{n1} & \dots & B_{nn}-\dfrac{(B_{1n}x_1,x_1)}{(B_{11}x_1,x_1)}B_{n1} \end{vmatrix}$$

is strongly positive or negative, say positive. Taking $B'_{jk}=B_{jk}-\dfrac{(B_{1k}x_1,x_1)}{(B_{11}x_1,x_1)}B_{j1},\ \ j=1,2,\dots,n,\ \ k=1,2,\dots,n$ we obtain

$$\Delta=\begin{vmatrix} B_{11} & B'_{12} \ \dots & B'_{1n} \\ \dots & \dots & \dots \\ B_{n1} & B'_{n2}\dots & B'_{nn} \end{vmatrix}\gg 0$$

and $\otimes\det(B'_{jk})\gg 0,\ \ j,k=2,\dots,n.$ That is without changing the notations for B_{jk} we suppose that (5) holds.

Now by the induction, for any fixed $\lambda_1\in R$, there exists a unique $\left(\lambda_2(\lambda_1),\dots,\lambda_n(\lambda_1)\right)\in R^{n-1}$ such that

$A_i-\lambda_1B_{i1}-\lambda_2B_{i2}-\dots-\lambda_nB_{in}\geq 0,\ \ i=2,3,\dots$ and

$(A_i-\lambda_1B_{i1}-\lambda_2B_{i2}-\dots-\lambda_nB_{in})x_i=0$ for some $0\neq x_i\in H_i,\ i=2,\dots,n.$
It follows from

$$\left(\begin{vmatrix} A_2-\lambda_1B_{21}-\lambda_2B_{22}-\dots-\lambda_nB_{2n} & B_{23}\dots & B_{2n} \\ \dots & \dots & \dots \\ A_n-\lambda_1B_{n1}-\lambda_2B_{n2}-\dots-\lambda_nB_{nn} & B_{n3}\dots & B_{nn} \end{vmatrix}(x_2\otimes\dots\otimes x_n),(x_2\otimes\dots\otimes x_n)\right)$$

$$\begin{vmatrix} \left((A_2-\lambda_1B_{21}-\lambda_2B_{22}-\dots-\lambda_nB_{2n})x_2,x_2\right) & (B_{23}x_2x_2) & \dots & (B_{2n}x_2,x_2) \\ \dots & \dots & & \dots \\ \left((A_n-\lambda_1B_{n1}-\lambda_2B_{n2}-\dots-\lambda_nB_{nn})x_n,x_n\right) & (B_{n3}x_n,x_n) & \dots & (B_{nn}x_n,x_n) \end{vmatrix}=0$$

that

$$\left(\begin{vmatrix} A_2 & B_{23}\dots & B_{2n} \\ \dots & \dots & \dots \\ A_n & B_{n3}\dots & B_{nn} \end{vmatrix}(x_2\otimes\dots\otimes x_n),(x_2\otimes\dots\otimes x_n)\right)$$

$$-\lambda_1\left(\begin{vmatrix} B_{21} & B_{23}\dots & B_{2n} \\ \dots & \dots & \dots \\ B_{n1} & B_{n3}\dots & B_{nn} \end{vmatrix}(x_2\otimes\dots\otimes x_n),(x_2\otimes\dots\otimes x_n)\right)$$

$$-\lambda_2\left(\begin{vmatrix} B_{22} & B_{23}\dots & B_{2n} \\ \dots & \dots & \dots \\ B_{n2} & B_{n3}\dots & B_{nn} \end{vmatrix}(x_2\otimes\dots\otimes x_n),(x_2\otimes\dots\otimes x_n)\right)=0$$

or

$$\left((\Gamma_2-\lambda_1\Delta_{12}+\lambda_2\Delta_{11})\bar{x},\bar{x}\right)=0,$$

where Δ_{1k} is the algebraic cofactor of B_{1k} in the determinant Δ, Γ_k is the algebraic cofactor of B_{1k} in the determinant

$$\Gamma=\begin{vmatrix} A_1 & B_{12}\dots & B_{1n} \\ \dots & \dots & \dots \\ A_n & B_{n2}\dots & B_{nn} \end{vmatrix}$$

and $\bar{x}=x_2\otimes\dots\otimes x_n$.. In like manner we have

$$\left((\Gamma_3-\lambda_1\Delta_{13}+\lambda_3\Delta_{11})\bar{x},\bar{x}\right)=0,\dots.$$
$$\left((\Gamma_n-\lambda_1\Delta_{1n}+\lambda_n\Delta_{11})\bar{x},\bar{x}\right)=0,$$

which give

$$\lambda_2(\lambda_1) = \frac{-(\Gamma_2 \bar{x}, \bar{x})}{(\Delta_{11} \bar{x}, \bar{x})} + \lambda_1 \frac{(\Delta_{12} \bar{x}, \bar{x})}{(\Delta_{11} \bar{x}, \bar{x})}, \ldots,$$

$$\lambda_n(\lambda_1) = \frac{-(\Gamma_n \bar{x}, \bar{x})}{(\Delta_{11} \bar{x}, \bar{x})} + \lambda_1 \frac{(\Delta_{1n} \bar{x}, \bar{x})}{(\Delta_{11} \bar{x}, \bar{x})}.$$

Consider the expression

$$\left((A_1 - \lambda_1 B_{11} - \lambda_2(\lambda_1) B_{12} - \ldots - \lambda_n(\lambda_1) B_{1n}) x_1, x_1\right)$$

$$= \left((A_1 - \lambda_1 B_{11}) x_1, x_1\right) - \lambda_2(\lambda_1)\left(B_{12} x_1, x_1\right) - \ldots - \lambda_n(\lambda_1)\left(B_{1n} x_1, x_1\right) = \left((A_1 - \lambda_1 B_{11}) x_1, x_1\right)$$

$$-\lambda_1 \left(B_{12} x_1, x_1\right) \frac{(\Delta_{12} \bar{x}, \bar{x})}{(\Delta_{11} \bar{x}, \bar{x})} + \left(B_{12} x_1, x_1\right) \frac{(\Gamma_2 \bar{x}, \bar{x})}{(\Delta_{11} \bar{x}, \bar{x})} - \ldots$$

$$-\lambda_1 \left(B_{1n} x_1, x_1\right) \frac{(\Delta_{1n} \bar{x}, \bar{x})}{(\Delta_{11} \bar{x}, \bar{x})} + \left(B_{1n} x_1, x_1\right) \frac{(\Gamma_n \bar{x}, \bar{x})}{(\Delta_{11} \bar{x}, \bar{x})}$$

$$= \frac{1}{(\Delta_{11} \bar{x}, \bar{x})} \left[\left(A_1 x_1, x_1\right)(\Delta_{11} \bar{x}, \bar{x}) + \left(B_{12} x_1, x_1\right)(\Gamma_2 \bar{x}, \bar{x}) + \ldots + \left(B_{1n} x_1, x_1\right)(\Gamma_n \bar{x}, \bar{x}) \right]$$

$$- \frac{\lambda_1}{(\Delta_{11} \bar{x}, \bar{x})} \left[\left(B_{11} x_1, x_1\right)(\Delta_{11} \bar{x}, \bar{x}) + \left(B_{12} x_1, x_1\right)(\Delta_{12} \bar{x}, \bar{x}) + \ldots + \left(B_{1n} x_1, x_1\right)(\Delta_{1n} \bar{x}, \bar{x}) \right]$$

$$= \frac{1}{(\Delta_{11} \bar{x}, \bar{x})} \left([\Gamma - \lambda_1 \Delta]\left(x_1 \otimes \bar{x}\right), \left(x_1 \otimes \bar{x}\right) \right).$$

Since $\Delta > 0$, there exists a unique $\lambda_1 \in R$ such that $\Gamma - \lambda_1 \Delta \geq 0$ and $Ker(\Gamma - \lambda_1 \Delta) \neq \{0\}$, which means that

$$A_1 - \lambda_1 B_{11} - \lambda_2(\lambda_1) B_{12} - \ldots - \lambda_n(\lambda_1) B_{1n} \geq 0 \text{ and}$$
$$Ker(A_1 - \lambda_1 B_{11} - \lambda_2(\lambda_1) B_{12} - \ldots - \lambda_n(\lambda_1) B_{1n}) \neq \{0\}.$$

Thus we have that the statement of the theorem holds for all n.

Now let us show that this theorem is the generalization of the main theorem of linear algebra on the existence and uniqueness of solution of the system (1) when $\det(b_{jk}) > 0$. If $H_1 = H_2 = \ldots = H_n = R$ the condition (3) together with (4) implies $\left(a_i - b_{i1}\lambda_1 - b_{i2}\lambda_2 - \ldots - b_{in}\lambda_n\right)x_i = 0$ for some $x_i \in R$, that is for all $x_i \in R$, in fact. Therefore the condition $\det(b_{jk}) > 0$ implies that the system of equations (1) has a unique solution. If $\det(b_{jk}) < 0$, multiplying one of rows by -1 again we have that the system (1) has a solution.

Let us note that the Theorem 1 can be applied to solve some problems of MPST and make some new observations. Atkinson [2] established that if the tensor determinant Δ is positive then for any choice of $\varepsilon_i = \pm 1,\ i = 1, 2, \ldots, n$ there exists ($\left(\lambda_1, \lambda_2, \ldots, \lambda_n\right) \in R^n$ such that

$$\varepsilon_i\left(\lambda_1 B_{i1} + \lambda_2 B_{i2} + \ldots + \lambda_n B_{in}\right) \gg 0,\ \ i = 1, 2, \ldots, n.$$

Now using the Theorem 1 we can prove the following

Theorem 2. If $\dim H_i < \infty,\ i = 1, 2, \ldots, n$, and $\Delta > 0$, then for any choice of $\varepsilon_i = \pm 1,\ i = 1, 2, \ldots, n$ and fixed $i_0 \in \{1, 2, \ldots, n\}$ there exists $\left(\lambda_1, \lambda_2, \ldots, \lambda_n\right) \in R^n$ such that

1) $\varepsilon_i\left(\lambda_1 B_{i1} + \lambda_2 B_{i2} + \ldots + \lambda_n B_{in}\right) \geq 0,\ \ i = 1, 2, \ldots, n$ and
2) $\varepsilon_{i_0}\left(\lambda_1 B_{i_0 1} + \lambda_2 B_{i_0 2} + \ldots + \lambda_n B_{i_0 n}\right) > 0$
3) $Ker\left(\lambda_1 B_{i1} + \lambda_2 B_{i2} + \ldots + \lambda_n B_{in}\right) \neq \{0\}$ for $i \neq i_0$.

Proof. For simplicity we consider the case $i_0 = 1$ and $\varepsilon_i = 1,\ \ i = 1, 2, \ldots, n$ As in case of the proof of Theorem we may suppose that (5) holds and then using the fact that there exists $\left(\lambda_2, \ldots, \lambda_n\right) \in R^{n-1}$ with

$$B_{i1} - \lambda_2 B_{i2} - \lambda_3 B_{i3} - \ldots - \lambda_n B_{in} \geq 0,\ \text{ and}$$
$$Ker(B_{i1} - \lambda_2 B_{i2} - \lambda_3 B_{i3} - \ldots - \lambda_n B_{in}) \neq \{0\},\ \ i = 2, 3, \ldots$$

we consider the determinant

$$\Delta = \begin{vmatrix} B_{11} - \lambda_2 B_{12} - \lambda_3 B_{13} - \ldots - \lambda_n B_{1n} & \ldots & B_{1n} \\ \ldots & \ldots & \ldots \\ B_{n1} - \lambda_2 B_{n2} - \lambda_3 B_{n3} - \ldots - \lambda_n B_{nn} & \ldots & B_{nn} \end{vmatrix}.$$

It is not difficult to see that $B_{11} - \lambda_2 B_{i2} - \lambda_3 B_{i3} - \ldots - \lambda_n B_{in} \gg 0$. Indeed, if

$$\left(\left(B_{11} - \lambda_2 B_{12} - \lambda_3 B_{13} - \ldots - \lambda_n B_{1n}\right) x_1, x_1\right) \leq 0,$$

for some $x_1 \in H_1$, taking $x_i \in Ker\left(B_i - \lambda_2 B_{i2} - \ldots - \lambda_n B_{in}\right)$, $i = 2, 3, \ldots, n$ we have

$$\left(\Delta\left(x_1 \otimes x_2 \otimes \ldots \otimes x_n\right), \left(x_1 \otimes x_2 \otimes \ldots \otimes x_n\right)\right)$$

$$= \left(\left(B_{11} - \lambda_2 B_{12} - \lambda_3 B_{13} - \ldots - \lambda_n B_{1n}\right) x_1, x_1\right)(\Delta_{11}\left(x_2 \otimes \ldots \otimes x_n\right), \left(x_2 \otimes \ldots \otimes x_n\right))$$
$$+\left(\left(B_{21} - \lambda_2 B_{22} - \lambda_3 B_{23} - \ldots - \lambda_n B_{2n}\right) x_2, x_2\right)(\Delta_{21}\left(x_1 \otimes x_3 \otimes \ldots \otimes x_n\right), \left(x_1 \otimes x_3 \otimes \ldots \otimes x_n\right)) + \ldots \leq 0,$$

where Δ_{ik} is the algebraic cofactor of B_{ik} in the determinant Δ, which contradicts the positivity of the Δ.

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