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The Equations of Motion of a Null Curve in Lightlike Cone of 3-Dimensonal Minkowski Space

Mehmet ERDOĞAN, Jeta ALO and Gülşen YILMAZ
Beykent University, Faculty of Science and Letters,
Department of Mathematics, Ayazağa/Şişli, İstanbul,TURKEY
e-mail: erdoganm@beykent.edu.tr

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Abstract.

In this article we investigate null curves in the lightlike cone of 3-dimensional Lorentz-Minkowski space and give some characterizations of these curves.We also obtain the equations of motion of a null curve in the lightlike cone.

Mathematics Subject Classificiation : 53B30, 53B50, 53C80

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Özet.

Bu makalede 3-boyutlu Lorentz-Minkowski uzayının ışık konisindeki ışıksal eğriler incelendi ve bazı karekterizasyonları verildi. Ayrıca ışık konisindeki bir ışıksal eğrinin hareket denklemleri elde edildi.

Anahtar kelimeler: *Işıksal eğri, ışık konisi, 3-boyutlu Lorentz-Minkowski uzayındaki bir ışıksal parçacığın hareket denklemleri*

1. Introduction

There exist three families of curves of a proper semi-Riemannian manifold M_i^n (that is the index i of the manifold satisfies $1 \leq i \leq \dim M - 1$) with a semi-Riemannian metric g called spacelike, timelike and null (lightlike) depending on their causal characters.When we study null curves it occurs some difficulties because the arc lentgh vanishes so that it is impossible to normalize the tangent vector in the usual manner. Therefore we introduce a new parameter called pseudo-arc which normalize the derivative of the tangent vector [1].

Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ be the vectors in a 3-dimensional Lorentz space $M_1^3 := E_1^3$. Then the scalar product of X and Y is defined by

$$g(X,Y) = < x, y > = x_1y_1 + x_2y_2 - x_3y_3, \tag{1.1}$$

which is called a Lorentz product. Furthermore, a Lorentz cross product $X \times Y$ is given by

$$X \times Y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \tag{1.2}$$

The pseudo- Riemannian lightlike cone in E_1^3 is defined by

$$C^2 = \{x \in E_1^3 : g(x,x) = 0\}. \tag{1.3}$$

Now, suppose that E_1^3 be the 3-dimensional Minkowski space and C^2 the lightlike cone in E_1^3. A vector $x \neq 0$ in E_1^3 is called spacelike, timelike or null (lightlike), if $g(x,x) > 0$, $g(x,x) < 0$, $g(x,x) = 0$, respectively. A frame field $\{e_1, e_2, e_3\}$ on E_1^3 is called an asymptotic orthonormal frame field , if

$$< e_1, e_2 > = < e_1, e_3 > = < e_2, e_2 > = < e_3, e_3 > = 0 , \ < e_1, e_1 > = < e_2, e_3 > = 1$$

A frame field $\{e_1, e_2, e_3\}$ on E_1^3 is called a pseudo orthonormal frame field , if

$$< e_1, e_1 > = < e_2, e_2 > = - < e_3, e_3 > = 1 , \ < e_1, e_2 > = < e_1, e_3 > = < e_2, e_3 > = 0.$$

Definition 1.1. A curve $\sigma(t)$ in E_1^3 is called a Cartan curve, if for all $t \in I$, the vector fields $\sigma(t), \ \sigma'(t), \ \sigma''(t)$ are linearly independent and the vector fields $\sigma(t), \ \sigma'(t), \ \sigma''(t), \ \sigma'''(t)$ are linearly dependent, where

$$\sigma^{(n)}(t) = \frac{d^{(n)}\sigma(t)}{dt^n}.$$

Let $\sigma : I \to C^2 \subset E_1^3$ be a null curve on the 3-dimensional Lorentzian space E_1^3. Since $< \sigma(t), \sigma(t) > = 0$ and $< \sigma(t), d\sigma(t) > = 0$, $d\sigma(t)$ is spacelike. Then the induced arc length s of the curve $\sigma(t)$ can be defined by $ds^2 = < d\sigma(t), d\sigma(t) >$. If we take the arc length s of the curve $\sigma(t)$ as the parameter and denote $\sigma(s) = \sigma(t(s))$ we have $\sigma' = \sigma'(s)$ and $< \sigma'(s), \sigma'(s) > = 1$, then s is called the pseudo-arc parameter. Hence, the curve $\sigma'(s)$ is a unit spacelike curve and $\sigma'(s) = \frac{d\sigma}{ds}$ is a spacelike unit tangent vector field of the curve $\sigma(s)$. Now, let us consider vector fields

$\sigma'(s) = X = X(s),\ Y = Y(s),\ \sigma(\mathrm{s}) = Z = Z(s)$ along the curve $\sigma(s)$ satisfying the conditions :

$< Y, Y > = < Z, Z > = < X, Z > = < X, Y > = 0,$

$< X, X > = < Y, Z > = 1$.

Therefore, for the vector fields above we have the following Frenet formulas:

$\nabla_X X = Z,\ \nabla_X Y = -\kappa(s)Z,\ \nabla_X Z = \kappa(s)Z - Y$.

A curve defined as above is called a *Cartan framed null curve*.

Moreover, if the functions $k_1(t)$ and $k_2(t)$ are positive constants on the curve $\sigma(t)$, then we call the curve a *Cartan framed null curve with positive constant curvatures*. In this definition, if $k_2(t) \equiv 0$, then the curve is named as a *generalized null cubic* [2,3]. On the other hand, it is well known that for any constant curvature functions $k_1(t)$ and $k_2(t)$ at a point *p* of the manifold there exists only one Cartan framed null curve with constant curvatures $k_1(t)$ and $k_2(t)$ passing through *p* with velocity vector $\sigma'(p) = X(p)$ and satisfying above conditions [4].

2. Lightlike cone C^2 of the Minkowski space M_1^3

First, we will consider the lightlike cone C^2 of the Minkowski space M_1^3 which is, indeed, a lightlike surface of M_1^3. As for any $p \in C^2$, T_pC^2 is a plane of the M_1^3, we consider

$$T_p(C^2)^{\perp} = \{V_p \in T_pE_1^3 : \overline{g}(V_p, W_p) = 0,\ \ \forall W_p \in T_pC^2\}, \qquad (2.1)$$

and

$$RadT_pC^2 = T_pC^2 \cap T_p(C^2)^{\perp}. \qquad (2.2)$$

For C^2 is the lightlike cone of the Minkowski space M_1^3, we have $RadT_pC^2 \neq \{0\}$ at any $p \in C^2$ and the semi-Riemannian metric $\overline{g}$ on M_1^3 induces on C^2 a symmetric tensor field $g_p(X_p, Y_p)$ for any $p \in C^2$. In this case we know that g has constant rank 1 on C^2 and $T(C^2)^{\perp} = \cup_{p \in C^2} T_p(C^2)^{\perp}$ is a distribution on C^2, see [4,5].

Now, let us consider a complementary vector bundle $S(TC^2)$ of $T(C^2)^{\perp}$ in TC^2, namely

$$TC^2 = S(TC^2) \perp (TC^2)^{\perp}. \qquad (2.3)$$

As each $S(TC^2)_p$ is a screen subspace of T_pC^2, we will call $S(TC^2)$ the *screen distribution* on C^2. Since C^2 is paracompact there always exist $S(TC^2)$. Moreover, for a lightlike vector space W any complementary subspace to $RadW$ is non-degenerate, so it follows that $S(TC^2)$ is a non-degenerate distribution. Thus along C^2 we have the following decomposition

$$TM_1^3\big|_{C^2} = S(TC^2) \perp S(TC^2)^{\perp}, \tag{2.4}$$

where $S(TC^2)^{\perp}$ is orthogonal complementary vector bundle to $S(TC^2)$ in $TM_1^3\big|_{C^2}$. On the other hand , since C^2 is a lighlike surface of M_1^3 then there exists a unique vector bundle $tr(TC^2)$ of rank 1 over C^2, such that for any non-zero section ξ of $(TC^2)^{\perp}$ on a coordinate neighborhood of C^2, there exists a unique section N of $tr(TC^2)$ defined on the coordinate neighborhood of C^2 and satisfying

$$\overline{g}(N,\xi)=1,\ \overline{g}(N,N)=\overline{g}(N,W)=0,\quad \forall W\in\Gamma(S(TC^2)). \tag{2.5}$$

Then we may show that the following section satisfies ;

$$N=\frac{1}{\overline{g}(\xi,V)}\left\{V-\frac{\overline{g}(V,V)}{2\overline{g}(\xi,V)}\xi\right\}, \tag{2.6}$$

where, $V\neq 0$ is a vector which belongs to the complementary vector bundle of $T(C^2)^{\perp}$ in $S(TC^2)^{\perp}$. Hence, it follows from (2.5) that $tr(TC^2)$ is a lightlike vector bundle such that $tr(TC^2)_p \cap T_pC^2=\{0\}$ for any $p\in C^2$. Moreover, from (2.3) and (2.4) we have the following decompositions of $TM_1^3\big|_{C^2}$:

$$\begin{aligned} TM_1^3\big|_{C^2} &= S(TC^2)\perp\left(T(C^2)^{\perp}\oplus tr(TC^2)\right)\\ &=\mathrm{TC}^2\oplus tr(TC^2)\end{aligned} \tag{2.7}$$

Therefore, for any screen distribution $S(TC^2)$ we have a unique $tr(TC^2)$ which is complementary vector bundle to TC^2 in $TM_1^3\big|_{C^2}$ and satisfies (2.5). $tr(TC^2)$ is called the *lightlike transversal vector bundle* of C^2 with respect to $S(TC^2)$. It is known that any screen distribution is non-degenerate of constant index, in particular, any screen distribution on a lightlike hypersurface of a Lorentz manifold is Riemannian, i.e., the induced metric on $S(TC^2)$ is positive definite. So, $S(TC^2)$ is also a Riemannian one.

Now , let us consider $x\in C^2 \subset M_1^3$ and $x=(x_1,x_2,x_3)\neq 0$. The lightlike cone C^2 of M_1^3 is given by the equation

$$x_3^{\ 2}-x_1^{\ 2}-x_2^{\ 2}=0$$

and it is a lightlike surface. In order to prove this, we define

$$D=\left\{(u_1,u_3)\in \mathbb{R}^2 :\ u_3^{\ 2}-u_1^{\ 2}>0\right\}$$

and the local immersion φ of C^2 from *D* into M_1^3 by

$$x_3=u_3,\ x_1=u_1,\ x_2=\sqrt{x_3^{\ 2}-x_1^{\ 2}}.$$

Thus, the tangent bundle TC^2 on $\varphi(D)$ is spanned by

$$\frac{\partial}{\partial u_3}=\frac{\partial}{\partial x_3}+\frac{x_3}{x_2}\frac{\partial}{\partial x_2} \text{ and } \frac{\partial}{\partial u_1}=\frac{\partial}{\partial x_1}-\frac{x_1}{x_2}\frac{\partial}{\partial x_2}.$$

Then, we may easily have that

$$\xi=x_1\frac{\partial}{\partial u_1}+x_3\frac{\partial}{\partial u_3}=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3} \tag{2.8}$$

is orthogonal to TC^2 at any point of $\varphi(D)$. Another local immersion of *D* into M_1^2 for points with $x_2<0$ is given by

$$x_3=u_3,\ x_1=u_1,\ x_2=-\sqrt{x_3^{\ 2}-x_1^{\ 2}}.$$

Then, ξ given by (2.8) is also orthogonal to TC_1^2 at all points of $\varphi(D)$. For each immersion ξ is given as in the (2.8), so $(TC^2)^{\perp}$ is globally spanned by the position vector field on C^2.

Next, we consider

$$N=\frac{1}{2x_3^{\ 2}}\left\{-x_3\frac{\partial}{\partial x_3}+x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right\} \tag{2.9}$$

globally defined on C^2. As $\bar{g}(N,N)=0$ and $\bar{g}(N,\xi)=1$, consider the lightlike transversal vector bundle $tr(TC^2)$ spanned by N.Then the corresponding screen distribution $S(TC^2)$ is represented by vector fields

$X=\sum_{i=1}^{2} X_i \frac{\partial}{\partial X_i}$, satisfying

$$x_3X_3=0,\ \ x_1X_1+x_2X_2=0 \tag{2.10}$$

at points of C^2 and the integral curves of ξ are open sets of lightlike rays of C^2.

By the investigations above, we see that C^2 is a lightlike hypersurface whose $T(C^2)^{\perp}$ is globally spanned by the position vector field.

Now, let the curve $\sigma(t)$ be a null curve in the lightlike cone C^2 in E_1^3 which preserves its causal character. Then, all its tangent vectors and all other vectors in C^2 are null . Thus, we have ξ given by equation (2.8). Furthermore, we may prove the following result for the lightlike cone.

Theorem 2.1 ,[4]. The lightlike cone C^2 of M_1^3 is a totally umbilical lighlike hypersurface.

Now, let us consider the lightlike cone C^2 of M_1^3 given by the equations

$$x_1=f_1(u,v),\ \ x_2=f_2(u,v),\ \ x_3=f_3(u,v),\ \ \text{rank}\left[\frac{\partial f_i}{\partial u_j}\right]=2,$$

where $1\le i,j\le 2$ and $u_1=u, u_2=v$ and define

$$D_1=\begin{vmatrix}\frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u}\\ \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v}\end{vmatrix},\ \ D_2=\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_3}{\partial u}\\ \frac{\partial f_1}{\partial v} & \frac{\partial f_3}{\partial v}\end{vmatrix},\ \ D_3=\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u}\\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v}\end{vmatrix},$$

where since C^2 is lightlike we have $D_3^{\,2}=D_1^{\,2}+D_2^{\,2}$. In this case the distribution $RadTC^2=(TC^2)^{\perp}$ is locally spanned by

$$\xi=D_3\frac{\partial}{\partial x_3}+D_1\frac{\partial}{\partial x_1}-D_2\frac{\partial}{\partial x_2}.$$

We may easily see that ξ belongs to $\Gamma\left((TC^2)^{\perp}\right)$ by taking the natural frame field as

$$\frac{\partial}{\partial u_i} = \frac{\partial f_j}{\partial u_i}\frac{\partial}{\partial x_j}, \quad i = 1, 2.$$

and then C^2 is lightlike if and only if $\overline{g}(\xi,\xi) = 0$, which is equivalent with $D_3^{\ 2} = D_1^{\ 2} + D_2^{\ 2}$.

If we define a local section of TM_1^3 by $V = -D_3 \frac{\partial}{\partial x_3}$, we get

$$\overline{g}(V,\xi) = D_3^{\ 2}.$$

On the other hand the vector bundle spanned by

$$N = \frac{1}{D_3^{\ 2}}\left\{V + \frac{1}{2}\xi\right\}$$

is called the canonical lightlike transversal vector bundle of C^2 and

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0.$$

Lets consider the tangent bundle of $\sigma(t)$ and denote it by $T\sigma$, then its normal bundle manifold is defined by

$$(T\sigma)^{\perp} = \left\{X \in \Gamma(C^2) : g(X,V) = 0\right\}, \quad V \equiv \frac{\partial}{\partial t} \tag{2.11}$$

and its dimension is $\dim(T\sigma)^{\perp} = 2$. Furthermore, null curves satisfy the followings:

1) $(T\sigma)^{\perp}$ is a null bundle subspace of $C^2 \subset E_1^3$.

2) $(T\sigma) \cap (T\sigma)^{\perp} = T\sigma \rightarrow T\sigma \oplus (T\sigma)^{\perp} \neq TE_1^3 \approx E_1^3$.

3. The equations of motion of null curves in the lightlike cone

In this section, we will investigate null curves in the lightlike cone $C^2 \subset E_1^3$. Now,

$$\sigma : I \rightarrow C^2 \subset E_1^3$$

be a null Cartan curve such that $\{\sigma', \sigma'', \sigma'''\}$ is positively oriented for all $s \in I$, s being the pseudo-arc parameter, with Cartan frame $\{\sigma' = X, Y, Z\}$, where σ, X, Y, Z satisfy

$\langle\sigma,\sigma\rangle = \langle X, X\rangle = \langle Z,Z\rangle = 0$ and $\langle Y,Y\rangle = 1$, $\langle X,Z\rangle = -1$ (3.1)
and therefore, $\langle\sigma, X\rangle = \langle\sigma,Y\rangle = \langle X,Y\rangle = \langle Y,Z\rangle = 0$. The Cartan equations are given by

$$\begin{aligned} X' &= Y, \\ Y' &= -kX + Z, \\ Z' &= -kY, \end{aligned} \tag{3.2}$$

where the prime sign denotes covariant derivative and *k* is the Cartan curvature of the curve. The fundamental theorem for null curves tells us that *k* determines completely the null curve parametrized by the pseudo-arc length parameter up to Lorentzian transformations. Then any local geometrical scalar defined along null curves can always be expressed as a function of its curvature and derivatives. For an arbitrary parameter *t* of the curve $\sigma(t)$, the cone curvature function *k* is given by, [6],

$$k(t) = \frac{\left\langle \frac{d\sigma}{dt}, \frac{d^2\sigma}{dt^2} \right\rangle^2 - \left\langle \frac{d^2\sigma}{dt^2}, \frac{d^2\sigma}{dt^2} \right\rangle \left\langle \frac{d\sigma}{dt}, \frac{d\sigma}{dt} \right\rangle}{2\left\langle \frac{d\sigma}{dt}, \frac{d\sigma}{dt} \right\rangle^5}. \tag{3.3}$$

In this part we consider mechanical systems with Lagrangians which linearly depend on the curvature of a null curve. This curvature function is sometimes called torsion since it is obtained from the third derivative of the relativistic null path. The space of elementary fields in this theory is the set Ω of null Cartan curves in the lightlike cone $C^2 \subset E_1^3$. For the sake of simplicity σ will also denote a variation of σ by null curves

$$\sigma = \sigma(s,\varepsilon) : [a,b]\times(-\delta,\delta) \to C^2 \subset E_1^3, \tag{3.4}$$

where $\sigma(s,0)$ is the reparametrization of σ. Associated with such a variation is the variational vector field $V(s) = V(s,0)$, where $V = V(s,\varepsilon) = \frac{\partial\sigma}{\partial\varepsilon}(s,\varepsilon)$. Let η be the differentiable function satisfying $\frac{\partial\sigma}{\partial s}(s,\varepsilon) = \eta(s,\varepsilon)L(s,\varepsilon)$. Then we will write down $\sigma(s,\varepsilon), k(s,\varepsilon), V(s,\varepsilon)$ for the corresponding pseudo-arc length parameter. We consider the action $F : \Omega \to \mathbb{R}$ given by

$$F(\sigma)=\int_{\sigma}(\lambda+\mu k(s))ds, \tag{3.5}$$

λ and μ both being constant. Describing of the simplest action of a particle is achieved when it is proportional to the pseudo-arc length parameter (i.e. $\mu=0$), which has been studied by Nersesian and Ramos in [7,8].

Now, a null curve σ is said to be a critical point of the action F when

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(\sigma_{\varepsilon})=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\int_{\sigma_{\varepsilon}}(\lambda+\mu k)ds=0 \tag{3.6}$$

for all variation throught null curves of the variation of σ. To compute the first-order variation of this action, along the elementary field space Ω, and so the field equations describing the dynamics of the particle, we use a standard argument involving some integrations by parts. Then by using the Cartan equations we have

$$F'(0)=\left[\Lambda\right]\Big|_a^b-c\int_a^b\langle V,(\mu k'''+3\mu kk'-\lambda k')X\rangle ds \tag{3.7}$$

where

$$\Lambda=-c\mu X(h)+2c\mu\langle\nabla_X^2 V,Z\rangle+c(\mu k+\lambda)\langle\nabla_X V,Y\rangle-c\langle V,\nabla_X((\mu k+\lambda)Y)\rangle$$
$$+2c\left(\frac{1}{2}\mu k''-\lambda k+2\mu Z\right)\langle X,V\rangle+2c\mu k\langle X,Z\rangle,$$

V standing for a generic variational vector field along σ and
$h=-\langle\nabla_X^2 V,Z\rangle,$ where ∇ is the Levi-Civita connection on E_1^3.

We take curves with the same endpoints and having the same Cartan frame in them, so that $\left[\Lambda\right]\Big|_a^b$ vanishes, [9]. Under these conditions, the first-order variation is

$$F'(0)=-c\int_a^b\langle V,(\mu k'''+3\mu kk'-\lambda k')X\rangle ds, \tag{3.8}$$

from which we obtain the following theorem

Theorem 3.1 The trajectory σ is in the null space of 3-dimensional spacetime E_1^3 if and only if Y, Z and k are well defined on the whole null space and the following differential equation is satisfied

$$\mu k'''+3\mu kk'-\lambda k'=0 \tag{3.9}$$

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