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TIMELIKE CURVES OF CONSTANT SLOPE IN MINKOWSKI SPACE E_1^4

Hüseyin KOCAYİĞİT, Mehmet ÖNDER
Celal Bayar University, Faculty of Science and Art, Department of Mathematics, 45047 Manisa, Turkey, e-mail: mehmet.onder@bayar.edu.tr

ABSTRACT

In this paper, we will give some characterizations for timelike curves of constant slope in Minkowski space-time E_1^4.

Key Words: *Timelike curve, timelike vector, Minkowski space-time, curve of constant slope.*

ÖZET

Bu çalışmada, E_1^4 Minkowski 4-uzayındaki sabit eğimli timelike eğriler için bazı karakterizasyonları vereceğiz.

1. INTRODUCTION

Minkowski space-time E_1^4 is an Euclidean space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of three causal characters; it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null(lightlike) if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $x(s)$ in E_1^4 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike). Also recall that the pseudo-norm of an arbitrary vector $v \in E_1^4$ is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore α is a unit vector if

$g(v,v)=\pm 1$. The velocity of the curve $x(s)$ is given by $\|x'(s)\|$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v,w)=0$.

Let $e=(0,0,0,1)\in E_1^4$. A timelike vector $v=(v_1,v_2,v_3,v_4)$ is called *future pointing* (resp. *past pointing*) if $g(v,e)<0$ (resp. $g(v,e)>0$). Thus, a timelike vector $v=(v_1,v_2,v_3,v_4)$ is future pointing if and only if $v_2^2+v_3^2+v_4^2<v_1^2$ and $v_1^2>0$. Let v be a future pointing (or past pointing) timelike unit vector, also y be a future pointing (or past pointing) timelike unit vector. If the angle between v and y is $\theta\geq 0$ then we have

$$g(v,y)=-\cosh\theta,$$

and $\theta\in IR$ is called *hyperbolic angle*. Let x be a spacelike unit vector. Then the angle $\theta\geq 0$ between x and y is given by

$$g(x,y)=\sinh\theta,$$

and $\theta\in IR$ is called *Lorentzian timelike angle*.

Denote by $\{T(s),N(s),B_1(s),B_2(s)\}$ the moving Frenet frame along the curve $x(s)$ in the space E_1^4. Then T, N, B_1, B_2 are the tangent, the principal normal, the first binormal and the second binormal fields, respectively. A timelike (spacelike) curve $x(s)$ is said to be parameterized by a pseudo-arclength parameter s, i.e. $g(x'(s),x'(s))=-1$ ($g(x'(s),x'(s))=1$). In particular, a null curve $x(s)$ in E_1^4 is parameterized by a pseudo-arclength parameter s, if $g(x''(s),x''(s))=1$ where pseudo-arclength function s is defined as follows in [1]

$$s=\int_0^t\left(g(x''(t),x''(t))\right)^{1/4}.$$

Let $x(s)$ be a curve in Minkowski space-time E_1^4, parameterized by arclength function of s. Then for the timelike curve $x(s)$ the following Frenet equations are given in [8]:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = -1,\ g(N,N) = g(B_1,B_1) = g(B_2,B_2) = 1.$$

Recall the functions $k_1 = k_1(s)$, $k_2 = k_2(s)$ and $k_3 = k_3(s)$ are called respectively, the first, the second and the third curvature of curve $x(s)$.

In Euclidean space E^4, a regular curve $x(s)$ is a curve of constant slope or a helix provided the unit tangent T of $x(s)$ has a constant angle θ with some fixed line l directed in the unit vector U; that is $g(T,U) = \cos\theta$. The condition for a curve to be a constant slope in Euclidean space E^4 is usually given in the form

$$\frac{k_1^2}{k_2^2} + \left[\frac{1}{k_3} \frac{d}{ds} \left(\frac{k_1}{k_2} \right) \right]^2 = \tan^2\theta = constant, \qquad (1)$$

where k_1, k_2 and k_3 are first, second and third curvatures of Euclidean curve $\alpha(s)$, respectively[3].

Clearly, (1) has a meaning only if k_1, k_2 and k_3 are nowhere zero, and it is only under this precondition that (1) is a necessary and sufficient condition for a curve of constant slope or helix.

In this study we give the timelike curves of constant slope whose unit tangent T has a constant hyperbolic angle θ with some fixed timelike line l directed in the timelike unit vector U.

2. TIMELIKE CURVES OF CONSTANT SLOPE

Theorem 2.1. A regular timelike curve $x(s)$ with curvatures $k_1 > 0$, $k_2 > 0$ and $k_3 \neq 0$ is a curve of constant slope if and only if the following condition is satisfied,

$$\left(\frac{k_1}{k_2}\right)^2+\frac{1}{k_3^2}\left(\left(\frac{k_1}{k_2}\right)'\right)^2=1-\operatorname{sech}^2\theta=constant\cdot$$

Proof. Let $x(s)$ be a timelike curve of constant slope. Then for a timelike unit vector U we have $g(T,U)=-\cosh\theta$. Differentiating this equation with respect to s and using the Frenet formulae we get

$$g(N,U)=0.$$

Therefore U is in the subspace $T-B_1-B_2$ and can be written as follows

$$U=\alpha T+\beta B_1+\gamma B_2, \tag{2}$$

where $\alpha=-\cosh\theta,\ \beta=\sinh\varphi$ and $\gamma=\sinh\psi$. In (2) α,β and γ are called as direction cosines of U. Since U is unit, we have

$$-\alpha^2+\beta^2+\gamma^2=-1. \tag{3}$$

The differentiation of (2) gives

$$(\alpha k_1-\beta k_2)N+\left(\frac{d\beta}{ds}-\gamma k_3\right)B_1+\left(\frac{d\gamma}{ds}+\beta k_3\right)B_2=0,$$

and this equation yields

$$\beta=\frac{k_1}{k_2}\alpha=-\frac{1}{k_3}\frac{d\gamma}{ds},\quad \frac{d\beta}{ds}=\gamma k_3. \tag{4}$$

Since

$$\frac{d\beta}{ds}=\gamma k_3 \text{ and } \frac{d\beta}{ds}=\frac{k_3'}{k_3^2}\frac{d\gamma}{ds}-\frac{1}{k_3}\frac{d^2\gamma}{ds^2}$$

we find the second order linear differential equation in γ given by

$$\frac{d^2\gamma}{ds^2}-\frac{k_3'}{k_3}\frac{d\gamma}{ds}+\gamma k_3^2=0. \tag{5}$$

If we change variables in the above equation as $t=\int_0^s k_3(s)ds$ then we get

$$\frac{d^2\gamma}{dt^2}+\gamma=0,$$

the solution of this equation is

$$\gamma=A\cos\int_0^s k_3(s)ds+B\sin\int_0^s k_3(s)ds, \tag{6}$$

where A and B are constants. From (4) and (6) we have

$$A\sin\int_0^s k_3(s)ds - B\cos\int_0^s k_3(s)ds = \frac{k_1}{k_2}\alpha = \beta ,$$

$$A\cos\int_0^s k_3(s)ds + B\sin\int_0^s k_3(s)ds = \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\alpha = \gamma .$$

From these equations it follows that

$$A = \alpha\left(\frac{k_1}{k_2}\sin\int_0^s k_3(s)ds + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\cos\int_0^s k_3(s)ds\right), \tag{7}$$

$$B = \alpha\left(-\frac{k_1}{k_2}\cos\int_0^s k_3(s)ds + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\sin\int_0^s k_3(s)ds\right), \tag{8}$$

Hence using (3), (7) and (8) we get

$$A^2 + B^2 = \left[\left(\frac{k_1}{k_2}\right)^2 + \frac{1}{k_3^2}\left(\left(\frac{k_1}{k_2}\right)'\right)^2\right]\cosh^2\theta = \cosh^2\theta - 1 ,$$

or

$$\left(\frac{k_1}{k_2}\right)^2 + \frac{1}{k_3^2}\left(\left(\frac{k_1}{k_2}\right)'\right)^2 = 1 - \mathrm{sech}^2\theta = constant . \tag{9}$$

Conversely, if the condition (9) is satisfied for a regular timelike curve we can always find a constant timelike vector U which makes a constant angle with the tangent of the curve.

Consider the timelike unit vector defined by

$$U = -\left[T + \frac{k_1}{k_2}B_1 + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' B_2\right]\cosh\theta .$$

By taking account of the differentiation of (9), differentiations of U gives that $\frac{dU}{ds}=0$, this means that U is a constant vector. Therefore the timelike curve $x(s)$ is a curve of constant slope.

Theorem 2.2. A regular timelike curve $x(s)$ in Minkowski space-time E_1^4 is a curve of constant slope if and only if there exists a C^2-function f such that

$$fk_3=\frac{d}{ds}\left(\frac{k_1}{k_2}\right), \qquad \frac{d}{ds}f(s)=-k_3\frac{k_1}{k_2}. \tag{11}$$

Proof. If the timelike curve $x(s)$ is a curve of constant slope, by Theorem 2.1 we write

$$\left(\frac{k_1}{k_2}\right)\frac{d}{ds}\left(\frac{k_1}{k_2}\right)+\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right]=0, \tag{12}$$

and hence

$$\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)=-\frac{\left(\frac{k_1}{k_2}\right)\frac{d}{ds}\left(\frac{k_1}{k_2}\right)}{\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right]}. \tag{13}$$

If we write

$$f(s)=-\frac{\left(\frac{k_1}{k_2}\right)\frac{d}{ds}\left(\frac{k_1}{k_2}\right)}{\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right]}, \tag{14}$$

then

$$f(s)k_3=\frac{d}{ds}\left(\frac{k_1}{k_2}\right). \tag{15}$$

From (12) it can be written

$$\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right]=-k_3\frac{k_1}{k_2}. \tag{16}$$

By using (15) and (16) we have

$$\frac{d}{ds}f(s) = -k_3\frac{k_1}{k_2}. \tag{17}$$

Conversely, let $f(s)k_3 = \dfrac{d}{ds}\left(\dfrac{k_1}{k_2}\right)$. If we define a timelike unit vector U by

$$U = -\left(\cosh\theta T + \frac{k_1}{k_2}\cosh\theta B_1 + f(s)\cosh\theta B_2\right) \tag{18}$$

since the unit tangent T of $x(s)$ has a constant angle θ with some fixed lines l directed in the unit timelike vector U, $x(s)$ is a curve of constant slope.

Theorem 2.3. A timelike curve $x(s)$ is a curve of constant slope in Minkowski space-time E_1^4 if and only if

$$\frac{k_1}{k_2} = A\cos\theta + B\sin\theta. \tag{19}$$

Proof. Suppose that timelike curve $x(s)$ is a curve of constant slope in Minkowski space-time E_1^4. Then the condition in Theorem 2.2 is satisfied. Let us define C^2-function θ and C^1-functions $m(s)$ and $n(s)$ by

$$\theta(s) = \int_0^s k_3(s)ds, \tag{20}$$

$$\left.\begin{aligned} m(s) &= \frac{k_1}{k_2}\cos\theta - f(s)\sin\theta,\\ n(s) &= \frac{k_1}{k_2}\sin\theta + f(s)\cos\theta. \end{aligned}\right\} \tag{21}$$

If we differentiate equations (21) with respect to s and take account of (20), (15) and (17) we find that $m' = 0$ and $n' = 0$. Therefore, $m(s) = A,\ n(s) = B$ where A, B are constants. Now substituting these in (20) and solving the resulting equations for k_1 / k_2, we get

$$\frac{k_1}{k_2} = A\cos\theta + B\sin\theta,$$

which is (19).

Finally assume that (19) holds. Then from the equations in (21) we get

$$f = -A\sin\theta + B\cos\theta,$$

which satisfies the conditions of Theorem 2.2. So, the timelike curve $x(s)$ is a curve of constant slope in Minkowski space-time E_1^4.

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