

## THE FINITE DIFFERENCES SCHEMES FOR FIRST ORDER NONLINEAR SYSTEM EQUATIONS IN A CLASS OF DISCONTINUOUS FUNCTIONS

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### ABSTRACT

In this paper a new finite differences schemes of the Cauchy and initial-boundary value problem, for the first order system differential equations which describe some conservation laws is suggested. At first an auxiliary problem which is equivalent to the main problem in certain sense is introduced. On the basis of the auxiliary problem the simple and economical algorithms from computational point of view are proposed.

### ÖZET

Makalede bazı korunma kanunlarını ifade eden birinci mertebeden bir boyutlu nonlineer kısmi türevli diferansiyel denklemler sistemi için süreksiz fonksiyonlar sınıfında yeni bir sonlu farklar yöntemi önerilmiştir. Esas probleme denk olan ve bazı avantajlara sahip yardımcı problem içerilmiş ve yardımcı problem kullanılarak sade ve ekonomik sayısal algoritmalar üretilmiştir.

*Key Words:* Conservation laws, Mathematical modeling of water displacement of oil, Finite difference scheme in a class of discontinuous functions

### 1. INTRODUCTION

As usually, let  $(x, t)$  be an element of the space  $R^4$ , here is  $x = (x_1, x_2, x_3)$  and  $t$  is time and  $\partial\Omega$  is the boundary surface of  $\Omega \subset R^3$ . It is known that any conservation law expresses by the formula

$$\int_{\Omega} p(x, t) \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \oint_{\partial\Omega} \vec{A}(x, t) \vec{n} ds dt + \int_{t_1}^{t_2} \int_{\Omega} g(x, t) dx dt = 0. \quad (1)$$

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Here  $p$  and  $g$  are columns with size 3,  $A$  is a vector field of which dimension  $3 \times 3$ ,  $\vec{n}$  is a normal vector at any points of  $\partial\Omega$ . The equation (1) shows that the variation of the set function  $P(\Omega, t) = \oint_{\Omega} p(x, t) dx$  in  $(t_1, t_2)$  time

interval, is equal to the variation of  $\vec{A}(x, t)$  flow which passes over the boundary surface of  $\Omega$  under source function  $g(x, t)$ . In general, as the function  $p$  imply a mass, impulse, moment of impulse, energy and etc.

It is known that the state of the system is defined by the speed, pressure and density functions, that are the functions  $p = p(u, x, t)$ ,  $g = g(u, x, t)$ , and  $A = A(x, t)$  dependent of  $u = u(u_1, u_2, u_3)$ . For the differentiable  $u(x, t)$  the equation (1) is equivalent to

$$\int_{t_1}^{t_2} \int_{\Omega} (p_t + \text{div}A + g) dx dt = 0. \quad (2)$$

Since the domain  $\Omega \times (t_1, t_2)$  is arbitrary from (2) we get

$$p_t + \text{div}A + g = 0. \quad (3)$$

The equation (3) can be rewritten in a divergence form as

$$p_t + \sum_{j=1}^3 \frac{\partial f_j}{\partial x_j} = 0.$$

In theories, under some assumptions the conservation laws form first order partial differential equations. Hyperbolic systems of first order conservation laws in some space variable has been investigated in [5], and developed an adequate theory of the initial value problem for system of conservation laws.

In this paper we will develop a special differences scheme for system differential equations of conservation laws in a class of discontinuous functions.

## 2. MAIN AND AUXILIARY PROBLEMS

In the limit of this paper we will investigate the case of  $n = 1$ . Let  $I = (-\infty, \infty)$  and  $R_+^2 = I \times [0, T)$ , and we consider the following system of equations

$$\frac{\partial \varphi_1(u, v)}{\partial t} + \frac{\partial f_1(u, v)}{\partial x} = 0, \quad (4)$$

$$\frac{\partial \varphi_2(u, v)}{\partial t} + \frac{\partial f_2(u, v)}{\partial x} = 0 . \quad (5)$$

Here  $u = u(x, t)$ ,  $v = v(x, t)$  are components of  $\vec{u}$ ,  $x$  is space coordinate and  $t$  is time,  $\varphi_i(u, v)$  and  $f_i(u, v)$ , ( $i = 1, 2$ ) are differentiable functions of any coordinates  $u$  and  $v$ .

Many different problems of physics are modeled by the system equations (4),(5). For example:

I. If  $\varphi_1(u, v) = u$ ,  $f_1(u, v) = uv$ ,  $\varphi_2(u, v) = uv$ ,  $f_2(u, v) = uv^2 + p(v)$  then the system equations (4),(5) represent one dimensional, time dependent constant pressure isentropic of incompressible fluids, [2], [13], [15].

II. If  $\varphi_1 = v$ ,  $f_1 = -u$ ,  $\varphi_2 = u$ ,  $f_2 = p(v)$ ,  $p(v) = kv^{-\gamma}$  ( $\gamma > 1, k > 0$  are constants) the system equation models the flow of one dimensional constant entropy incompressible fluids, [2], [13], [15].

III. If  $\varphi_1(h, u) = h$ ,  $f_1(h, u) = uh$ ,  $\varphi_2(h, u) = u$ ,  $f_2(h, u) = \frac{u^2}{2} + gh$ , then the system of equations (4),(5) describing shallow water flows in an isolated ridge [3], [4], [14].

The initial-boundary value problem for the shallow water equations is investigated in [12]. It has been shown that the solution of this problem has the points of discontinuity of whose the locations unknown beforehand. Since the system (4),(5) are nonlinear, to obtain the exact solution of the Cauchy problem for (4),(5) is impossible, and besides, the Cauchy problem for (4),(5) has not the classical solution.

In this situation, it is necessary to develop a numerical method for obtaining the approximate solution with higher resolvability character. We will investigate the system equations (4),(5) under following initial

$$u(x, 0) = u_0(x), \quad (6)$$

$$v(x, 0) = v_0(x) \quad (7)$$

conditions. Here, the functions  $u_0(x)$  and  $v_0(x)$  are given functions having compact supports. Later we will call the problem (4)-(7) as the main problem.

### 2.1. Auxiliary Problem for Cauchy Problem

In this section we will develop a special numerical method for the problem (4)-(7) in a class of discontinuous functions.

The functions  $\varphi_i(u, v)$  and  $f_i(u, v)$  ( $i = 1, 2$ ) satisfy the following conditions:

1.  $\varphi_i(u, v)$  and  $f_i(u, v)$ , ( $i = 1, 2$ ) are continuous functions for both arguments  $u$  and  $v$ ;

2.  $\varphi_i(u, v)$  are differentiable functions with respect to  $u$  and  $v$ , and  $\frac{\partial \varphi_i}{\partial u}$  and  $\frac{\partial \varphi_i}{\partial v}$  do not vanish, the jacobians  $\frac{D(\varphi_1, \varphi_2)}{D(u, v)} \neq 0$ ,  $\frac{D(f_1, f_2)}{D(u, v)} \neq 0$ ;

3.  $f_i$ , ( $i = 1, 2$ ) are differentiable and  $f_{iu}''$  and  $f_{iv}''$  may change sign.

The weak solutions of the problem (4)-(7) are defined as:

**Definition 1.** The functions  $u(x, t)$  and  $v(x, t)$  satisfying the initial conditions (6) and (7) are called the weak solution of the problem (4)-(7) if the following integral relations

$$\iint_D \{\varphi_i(x, t)\varphi_1(u, v) + f_1(u, v)\varphi_x\} dxdt + \int_{-\infty}^{\infty} \varphi_1(u_0, v_0)\varphi(x, 0) dx = 0 \quad (8)$$

$$\iint_D \{\varphi_i(x, t)\varphi_2(u, v) + f_2(u, v)\varphi_x\} dxdt + \int_{-\infty}^{\infty} \varphi_2(u_0, v_0)\varphi(x, 0) dx = 0 \quad (9)$$

hold for every any test functions  $\varphi(x, t)$  which vanishes for sufficiently large  $t + |x|$ .

As it is seen in (8) and (9), the functions  $u(x, t)$  and  $v(x, t)$ , (even the functions  $\varphi_i(u, v)$  and  $f_i(u, v)$ , ( $i = 1, 2$ )) may be discontinuous, too.

In order to find the weak solutions of the problem (4)-(7), according to [8], [9], the following auxiliary problem

$$\frac{\partial p(x, t)}{\partial t} + f_1(x, t) = 0, \quad (10)$$

$$\frac{\partial q(x, t)}{\partial t} + f_2(x, t) = 0, \quad (11)$$

$$p(x, 0) = p_0(x), \quad (12)$$

$$q(x, 0) = q_0(x) \quad (13)$$

is introduced. Here the functions  $p_0(x)$  and  $q_0(x)$  are any differentiable solution of the equations respectively

$$\frac{dp_0(x)}{dx} = u_0(x) , \quad \frac{dq_0(x)}{dx} = v_0(x).$$

**Theorem 1.** *If the functions  $p(x,t)$  and  $q(x,t)$  are the solutions of the auxiliary problem (10)-(13), then the functions  $u(x,t)$  and  $v(x,t)$  defined by*

$$\begin{aligned} \varphi_1(u(x,t), v(x,t)) &= \frac{\partial p(x,t)}{\partial x}, \\ \varphi_2(u(x,t), v(x,t)) &= \frac{\partial q(x,t)}{\partial x} \end{aligned}$$

*are weak solutions of the main problem.*

### 2.2. Auxiliary Problem for Initial-Boundary Value Problem

In this section we will investigate the system equations (4),(5) by the initial conditions (6),(7) and the following boundary

$$u(0,t) = u_1(t), \quad v(0,t) = v_1(t) \quad (14)$$

conditions. Here  $u_1(t)$  and  $v_1(t)$  are given functions.

In this case the auxiliary problem for (4),(5) is

$$\frac{\partial}{\partial t} \int_0^x \varphi_1(u(\xi,t), v(\xi,t)) d\xi = f_1(u_1(t), v_1(t)) - f_1(u(x,t), v(x,t)), \quad (15)$$

$$\frac{\partial}{\partial t} \int_0^x \varphi_2(u(\xi,t), v(\xi,t)) d\xi = f_2(u_1(t), v_1(t)) - f_2(u(x,t), v(x,t)). \quad (16)$$

The initial conditions for (15),(16) are given in (6),(7). The problem (15),(16) and (14) is called as the second type auxiliary problem, [9], [10].

## 3. FINITE DIFFERENCES METHOD IN A CLASS OF DISCONTINUOUS FUNCTIONS

In literature, there are many papers that devoted to study numerical methods for the system equations of conservation laws. In this work, homogenous algorithms (not considering the existence of jump points) for the solutions of equation (4),(6) with suitable initial and boundary conditions are examined [6], [7]. In [11], the difference scheme for solving nonlinear system of equation of gas dynamic problem in a class of discontinuous functions has been studied.

### 3.1 Numerical Algorithm for Cauchy Problem

In order to develop the numerical method, the domain  $R_+^2$  is covered by the grid

$$\Omega_{h,\tau} = \{(x_i, t_k), x_i = ih, t_k = k\tau, dx > 0; i = \dots, -n, -n+1, \dots, -1, 0, 1, \dots, n, \dots; k = 0, 1, \dots\}$$

here  $h > 0$  and  $\tau > 0$  are the steps of the grid with respect to  $x$  and  $t$ , respectively. At any points  $(x_i, t_k)$  we approximate the problem (10)-(13) the following scheme

$$\wp_{i,k+1} = \wp_{i,k} + \tau f_1(U_{i,k}, V_{i,k}), \quad (17)$$

$$Q_{i,k+1} = Q_{i,k} + \tau f_2(U_{i,k}, V_{i,k}), \quad (18)$$

$$\wp_{i,0} = p_0(x_i), \quad (19)$$

$$Q_{i,0} = q_0(x_i). \quad (20)$$

The finite differences scheme (17)-(20) is an explicit, and it is known that for sufficiently small  $\tau$ , this scheme gives correct results.

For the problem (10)-(13) we can write the implicit scheme as

$$\wp_{i,k+1} = \wp_{i,k} - \tau f_1(U_{i,k+1}, \wp_{i,k+1}), \quad (21)$$

$$Q_{i,k+1} = Q_{i,k} - \tau f_2(U_{i,k+1}, V_{i,k+1}). \quad (22)$$

The initial conditions for (21),(22) are given in (19) and (20). Using the Newton iteration method we can obtain the solution of the nonlinear algebraic equations (21),(22). Here  $\wp_{i,k}$ ,  $Q_{i,k}$ ,  $U_{i,k}$ , and  $V_{i,k}$  denote the approximate values of the unknown functions  $p(x, t)$ ,  $q(x, t)$ ,  $u(x, t)$ , and  $v(x, t)$  at the point  $(x_i, t_k) \in \Omega_{h,\tau}$ .

**Theorem 2.** *If  $\wp_{i,k+1}$  and  $Q_{i,k+1}$  are the numerical solution of the problem (17)-(20), then the functions  $U_{i,k+1}$  and  $V_{i,k+1}$  obtained by*

$$\varphi_1(U_{i,k+1}, V_{i,k+1}) = \wp_{\bar{x}}, \quad (23)$$

$$\varphi_2(U_{i,k+1}, V_{i,k+1}) = Q_{\bar{x}} \quad (24)$$

*are the numerical solutions of the main problem.*

Applying the Runge-Kutta method to the problem (21),(22), we can get a higher order finite-difference scheme for the main problem with respect to  $\tau$ .

### 3.2. Numerical Algorithm for Initial-Boundary Value Problem

In order to approximate the problem (15),(16) and (14) by finite differences scheme at any points  $(x_i, t_k)$  of the grid  $\Omega_{h,\tau}$ , we will use the following quadrature

$$\int_0^x f(\xi, t) d\xi = h \sum_{j=1}^i F_{j,k} \quad (25)$$

formula. Taking (25) into consideration, the system equations (15) and (16) are approximated a

$$\varphi'_{1\tilde{u}}(\tilde{U}_{i,k}, V_{i,k+1})U_{i,k+1} + \varphi'_{1\tilde{v}}(U_{i,k}, \tilde{V}_{i,k})V_{i,k+1} = \bar{A}_1 + \varphi'_{1\tilde{u}}U_{i,k} + \varphi'_{1\tilde{v}}V_{i,k}, \quad (26)$$

$$\varphi'_{2\tilde{u}}(\tilde{U}_{i,k}, V_{i,k+1})U_{i,k+1} + \varphi'_{2\tilde{v}}(U_{i,k}, \tilde{V}_{i,k})V_{i,k+1} = \bar{A}_2 + \varphi'_{2\tilde{u}}U_{i,k} + \varphi'_{2\tilde{v}}V_{i,k}, \quad (27)$$

where

$$\begin{aligned} \bar{A}_\nu &= \frac{\tau}{h} f_\nu(u_1(t_k), v_1(t_k)) - \frac{\tau}{h} f_\nu(U_{i,k}, V_{i,k}) \\ &- \sum_{j=1}^{i-1} [\varphi_\nu(U_{j,k+1}, V_{j,k+1}) - \varphi_\nu(U_{j,k}, V_{j,k})]; \end{aligned} \quad (28)$$

( $\nu = 1, 2$ ) and  $\varphi'_{\nu\tilde{u}}$  and  $\varphi'_{\nu\tilde{v}}$  are

$$\begin{aligned} \varphi'_{1\tilde{u}}(\tilde{U}_{i,k}, V_{i,k+1}) &= \frac{\varphi_1(U_{i,k+1}, V_{i,k+1}) - \varphi_1(U_{i,k}, V_{i,k+1})}{U_{i,k+1} - U_{i,k}}, \\ \varphi'_{2\tilde{v}}(\tilde{U}_{i,k}, V_{i,k+1}) &= \frac{\varphi_2(U_{i,k}, V_{i,k+1}) - \varphi_2(U_{i,k}, V_{i,k})}{V_{i,k+1} - V_{i,k}}, \end{aligned}$$

here,  $\tilde{U}_{i,k} \in (U_{i,k}, U_{i,k+1})$ ,  $\tilde{V}_{i,k} \in (V_{i,k}, V_{i,k+1})$ . Since the jacobian  $\frac{D(\varphi_1, \varphi_2)}{D(u, v)} = 0$ , is not equal to zero we can find the explicit expressions for

$U_{i,k+1}$  and  $V_{i,k+1}$  from (26),(27), respectively.

#### 4. APPLICATION THE SUGGESTED METHOD TO THE PROBLEM OF OIL DISPLACEMENT WITH CHEMICAL ACTIVE MIXTURE

In order to demonstrate the efficiency of the suggested method we have applied this method to the problem which describes the process displacement of oil by the chemical active solvents (surfactant polymer, etc.)

It is known that in the state of thermodynamic equilibrium, in which capillary pressure is neglected, then one dimensional system equations that describe the large scale process of oil-water displacement, takes the following form

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$$m \frac{\partial \sigma}{\partial t} + w(t) \frac{\partial F_1(\sigma, c)}{\partial x} = 0, \quad (29)$$

$$m \frac{\partial G(\sigma, c)}{\partial t} + w(t) \frac{\partial F_2(\sigma, c)}{\partial x} = 0. \quad (30)$$

Here  $F_1(\sigma, c)$ ,  $F_2(\sigma, c)$  and  $G(\sigma, c)$  are given functions as

$$F_1(\sigma, c) = \frac{K_w(\sigma, c)}{K_w(\sigma, c) + \mu K_0(\sigma, c)},$$

$$F_2(\sigma, c) = F_1(\sigma, c)c + [1 - F_1(\sigma, c)]\varphi(c),$$

$$G(\sigma, c) = \sigma c + (1 - \sigma)\varphi(c) + \frac{a(v)}{m}, \quad \mu = \frac{\mu_w(c)}{\mu_0(c)},$$

$$\mu_w(c) = 1 + 0.5 \frac{c}{c^*} - 0.3 \left( \frac{c}{c^*} \right)^2, \quad c^* = 0.086,$$

$$\mu_0(c) = 14.5 - 14.7 \frac{c}{c^*} + 5.5 \left( \frac{c}{c^*} \right)^2.$$

The Buckley-Leverett's functions are expressed as

$$K_w(\sigma, c) = \begin{cases} 0, & \sigma < 0.2 \\ \left( \frac{\sigma - 0.2}{0.8} \right)^3, & \sigma \geq 0.2, \end{cases}$$

$$K_0(\sigma, c) = \begin{cases} 0, & \sigma \geq 0.83 \\ K_2^1 C_2, & 0.74 < \sigma < 0.83 \\ (K_2^1 - K_2^2)C_2 + K_2^2, & \sigma \leq 0.74, \end{cases}$$

$$C_2 = \frac{\varphi(c)}{Kc^*}, \quad \varphi(c) = Kc, \quad K = 2, \quad a(\sigma) = 0,$$

$$K_2^1 = \left( \frac{0.83 - \sigma}{0.834} \right)^3, \quad K_2^2 = \left( \frac{0.74 - \sigma}{0.715} \right)^3,$$

$$m = 0.2, \quad w(t) = 3.016 \times 10^{-5}.$$

Here  $K_w(\sigma, c)$ ,  $K_0(\sigma, c)$  denote partial permeability of water and oil phases respectively;  $F_1(\sigma, c)$ ,  $F_2(\sigma, c)$ , which determine water and oil fractions in the total volume of fluid of the mixture are called Buckley-



Leverett's functions;  $a(c)$  – the mass of absorption of active solution in per unit volume of the porous medium;  $\varphi(c)$  – the mass concentration of the active solution in the water and oil phase;  $\mu_w(c)$ ,  $\mu_o(c)$  dynamical viscosity of water and oil respectively;  $m$  – porosity;  $\sigma$  – saturation of water phase;  $c$  – concentration of mixture in water phase.

The initial and boundary conditions for (29),(30) are

$$\sigma(x,0) \equiv \sigma_0(x) = 0.2, \quad c(x,0) \equiv c_0(x) = 0, \quad (31)$$

$$\sigma(0,t) \equiv \sigma_1(t) = 0, \quad c(0,t) \equiv c_1(t) = \begin{cases} 1, & 0 < t \leq T_0 \\ 0, & t > T_0. \end{cases} \quad (32)$$

Here,  $T_0$  is the pumping time of chemical active water into the porous medium.

In order to approximate the problem (29)-(32) at first, we introduce the second type auxiliary problem as

$$m \int_0^x \sigma(\xi, t) d\xi + w(t)F_1(\sigma(x, t), c(x, t)) = w(t)F_1(\sigma_1(t), c_1(t)) \quad (33)$$

$$m \int_0^x G(\sigma(\xi, t), c(\xi, t)) d\xi + w(t)F_2(\sigma(x, t), c(x, t)) = w(t)F_2(\sigma_1(t), c_1(t)). \quad (34)$$

Using (25), the system equations (33),(34) are approximated as

$$\begin{aligned} \sum_{i,k+1} &= \sum_{i,k} + \frac{\tau}{h} w(t_k) F_1(\sigma_1(t_k), c_1(t_k)) - \\ &- \frac{\tau}{h} w(t_k) F_1(\sum_{i,k}, C_{i,k}) - \sum_{j=1}^{i-1} (\sum_{j,k+1} - \sum_{j,k}), \end{aligned} \quad (35)$$

$$\begin{aligned} C_{i,k+1} &= C_{i,k} + [G'_{\tilde{c}}]^{-1} \\ &\left[ \frac{\tau}{h} w(t_k) F_2(\sigma_1(t_k), c_1(t_k)) - \frac{\tau}{h} w(t_k) F_2(\sum_{i,k}, C_{i,k}) - \right. \\ &\left. - \sum_{j=1}^{i-1} [G(\sum_{j,k+1}, C_{j,k+1}) - G(\sum_{j,k}, C_{j,k})] - G'_{\tilde{c}} (\sum_{i,k+1} - \sum_{i,k}) \right]. \end{aligned} \quad (36)$$

Here  $G'_{\tilde{c}}$  is the value of  $\frac{\partial G}{\partial \sigma}$  at the time layer  $t = t_k$ . Using the difference scheme (35),(36) computer experiments has been carried out and the results of the experiments are demonstrated in Figure 1 and Figure 2.

### 5. CONCLUSION

In this paper an original numerical method for finding the weak solution of the initial and initial-boundary value problems for first order system of nonlinear partial equations in a class of discontinuous functions is developed. Suggested method has been applied to the displacement of oil process by the chemical active mixture in porous medium.

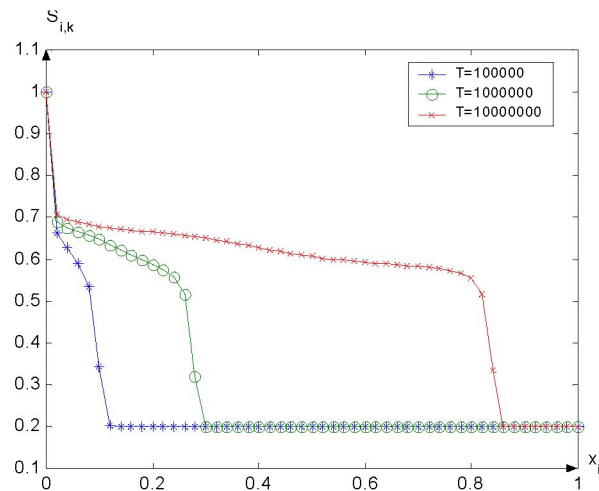


Figure 1. Dynamical distribution of water saturation

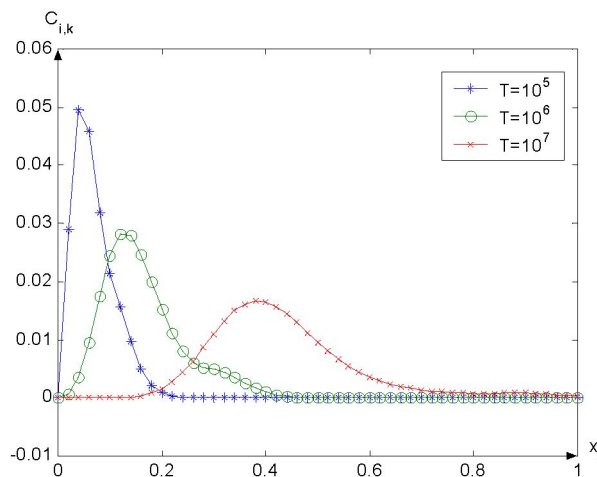


Figure 2. Dynamical distribution of concentration

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