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On Existence of Solution of a Nonlocal Boundary Value Problem for Heat Transfer Equation

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Abstract

In this work, it is considered non–homegenous heat equation with nonlocal boundary conditions. Non-local boundary conditions are reduced to pointwise non classical boundary value problem. It is applied the metod of separation variables to this problem. It is found eigenvalues and eigenfunctions of Sturm-Liouville problem which appears in the solution, the expansion according to eigenfunctions is obtained. It is shown the existence of solution of the boundary value problem which is consider. This kind of boundary value problems appears on heat variation, diffusion and economy problems.

Keywords: *Heat equation, non-local boundary value problem, integral condition, Fourier method, adjoint boundary value problem, biorthogonal adjoint sequence.*

Isı İletimi Denklemi İçin Lokal Olmayan Bir Sınır Değer Probleminin Çözümünün Varlığı Üzerine

Özet

Çalışmada homojen olmayan ısı iletimi denklemi için lokal olmayan sınır koşulları göz önüne alınır. Lokal olmayan sınır koşulları noktasal klasik olmayan sınır değer problemine indirgenir ve değişkenlere ayırarak problemin çözümü bulunur. Çözümün varlığı gösterilirken karşılaşılan Sturm-Liouville probleminin özdeğer ve özfonksiyonları bulunur, özfonksiyonlara göre ayrışımı elde edilir. Bu biçimde sınır değer problemleri ile ısı değişimi, difüzyon olaylarında, ekonomi problemlerinde karşılaşılabilir.

1. Introduction

Conditions for the problems related to the heat transfer, vibration of elastic string and economics can be given not only at the end points but also in the form of integral [1,2,3,6]. This form of the condition shows that the physical phenomena is related not only to end points but also to whole domain. For example, heat distribution on the thin heated string is expressed by the following boundary condition

$$\int_0^1 u(x,t)dx = \mu(t)$$

where $u(x,t)$ represents the heat at point x at time t and $\mu(t)$ determines the rule of variation at the amount of total heat on the string [5,7,12].

When the condition is given in the integral form, in order to make easier the solution of the problem, it is important to reduce the boundary conditions to the discrete form. The methods of the mathematical physics can be applied to the obtained mixed boundary value problem. In this study firstly, the boundary conditions are found at the different points of the string for the resulting inhomogeneous heat transfer equation. Then, separation of variables and Fourier method are applied by using the linearity of the problem. Similar problems were studied in [1,3,6,11,16] by the different methods.

After the separation of the variables we encounter the boundary value problem for the second order ordinary differential equation. For the existence of the solution, it is necessary to investigate the nonclassical and non selfadjoint boundary value problem. In this case, for the expansion of the function given in the initial condition to the eigenfunction, the results of [9,14,15] and the methods of [5] are used. For the further steps we introduce the theorem we will use.

Theorem 1:[9,14**]** If regular boundary conditions that constitute non selfadjoint L linear differential operator are stong regular, (if the order of the operator is even) the eigenfunctions of the L operator constitute Riesz base in the space $L_2(0,1)$

A similar result which is related to expansion has been obtained in [7,9,13,14] with a different method for non strong regular case.

2. Motivation

In this study in domain $D = \{(x,t): 0 < x < 1, 0 < t < T\}$ we consider the below nonhomogeneous heat transfer equation

$$\frac{\partial \upsilon}{\partial t} = \frac{\partial^2 \upsilon}{\partial x^2} + F(x,t) \tag{2.1}$$

with the boundary conditions

$$\upsilon_x(0,t)=v(t),\ 0\le t\le T \tag{2.2}$$

$$\int_0^1 x\upsilon(x,t)dx=\mu(t),\ 0\le t\le T \tag{2.3}$$

and initial conditions:

$$\upsilon(x,0)=\upsilon_0(x),\ \ 0\le x\le 1 \tag{2.4}$$

Our purpose is to find out the solution of equation (2.1) that is continuous in the region $\overline{D}=\{(x,t)|0\le x\le 1,\ 0\le t\le T\}$ and satisfying the conditions (2.2)-(2.4).

Suppose that $F(x,t)$, $v(t)$, $\mu(t)$, $\upsilon_0(x)$ are defined, continuously differentiable functions in a given domain D.

We first introduce the classical solution for the boundary value problem (2.1)-(2.4).

2.1.1. Definition: The solution which satisfies the following conditions is called the classical solution of the boundary value problem (2.1)-(2.4). This solution;

1) is continious in domain $\overline{D}$,

2) has first order derivative with respect to *t* and second order derivative with respect to *x*.

3) satisfies the equation (2.1) and conditions (2.2)-(2.4) in the mean of classical.

For the existence of a such solution, it is assumed that the consistency conditions

$$\int_0^1 x\upsilon_0(x)dx=\mu(0) \text{ and } \frac{d\upsilon_0(0)}{dx}=v(0) \tag{2.5}$$

holds.

In order to investigate boundary value problem (2.1)-(2.4), firstly the conditions (2.2), (2.4) are reduced to homogeneous form. By considering that the $\mu(t)$ and $\nu(t)$ functions are differentiable we define a new transformation

$$\upsilon(x,t)=u(x,t)+V(x,t) \tag{2.6}$$

where

$$V(x,t)=\frac{1}{3}(3x-2)\nu(t)+4x^2\mu(t) \tag{2.7}$$

Thus we obtain

$$\upsilon_x(0,t)=u_x(0,t)+V_x(0,t)$$

or due to the equation (2.2) we get

$$\nu(t)=u_x(0,t)+\nu(t) \text{ and } u_x(0,t)=0$$

By substituting the equation (2.6) into the equation (2.3) it is found

$$\int_0^1 x\upsilon(x,t)dx=\int_0^1 xu(x,t)dx+\int_0^1 xV(x,t)dx$$

or

$$\int_0^1 xu(x,t)dx=0$$

Taking into account them for the function $u(x,t)$ on this $\overline{D}$ region it is obtained the following boundary value problem:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(x,t), \tag{2.8}$$

$$u_x(0,t)=0, \tag{2.9}$$

$$\int_0^1 xu(x,t)dx=0, \tag{2.10}$$

$$u(x,0)=\varphi(x), \tag{2.11}$$

where

$$f(x,t)=F(x,t)+8\mu(t)+\frac{1}{3}(2-3x)\frac{\partial v(t)}{\partial t}-4x^2\frac{\partial \mu(t)}{\partial t},$$

$$\varphi(x)=\upsilon_0(x)+\frac{1}{3}(2-3x)v(0)-4x^2\mu(0).$$

According to reconciliation condition (2.5), it is found $\int_0^1 x\varphi(x)dx=0$ and $\left.\frac{d\varphi}{dx}\right|_{x=0}=0.$

Since (2.8)-(2.11) is linear problem, then the solution of boundary value problem (2.8)-(2.11) can be shown in the form of the total of the solutions to following two boundary value problems ($f(x,t)=0,\varphi(x)\neq 0$, $f(x,t)\neq 0,\varphi(x)=0$):

I.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ (x,t)\in D \tag{2.12}$$

$$u_x(0,t)=0,\ 0\le t\le T \tag{2.13}$$

$$\int_0^1 xu(x,t)dx=0,\ 0\le t\le T \tag{2.14}$$

$$u(x,0)=\varphi(x),\ 0\le x\le 1 \tag{2.15}$$

II.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(x,t) \tag{2.16}$$

$$u_x(0,t)=0,\ 0\le t\le T \tag{2.17}$$

$$\int_0^1 xu(x,t)dx=0,\ 0\le t\le T \tag{2.18}$$

$$u(x,0)=0,\ 0\le x\le 1 \tag{2.19}$$

Thus, if the solution of I. the boundary value problem is $u_1(x,t)$, of II. boundary value problem is $u_2(x,t)$, then the solution of boundary value problem (2.8)-(2.11) is found as $u(x,t)=u_1(x,t)+u_2(x,t)$. Here, solution of II. boundary value problem which is generated by nonhomogeneous equation (2.16) can be found by expanding to the eigenfunctions of I. boundary value problem which is generated appropriate homogeneous equation (2.12).

By using partial integration in equation (2.12), in the form (2.14)

$$u_x(1,t)+u(0,t)-u(1,t)=0$$

it is obtained nonclassical boundary condition which is given at the two ends. Hence boundary value problem (2.12)-(2.15) transforms into an equivalent mixed boundary value problem as follows

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},(x,t)\in D \tag{2.20}$$

$$u_x(0,t)=0,\ 0\le t\le T \tag{2.21}$$

$$u_x(1,t)+u(0,t)-u(1,t)=0,\ 0\le t\le T \tag{2.22}$$

$$u(x,0)=\varphi(x),\ 0\le x\le 1 \tag{2.23}$$

3. The Method of Separation of Variables for Sturm-Liouville Problem

Let us consider the boundary value problem (2.20)-(2.23) which is equivalent to boundary value problem I. Here we defined the function $\varphi(x)$ in earlier chapter. To solve this problem we use Fourier method or the method of separation of variables. We will seek the non-zero solution of boundary value problem (2.20)-(2.23) in the form $u(x,t)=X(x)T(t)$.

As a result the ordinary differential equations

$$T'(t)+\lambda T(t)=0 \tag{3.1}$$

$$X''(x)+\lambda X(x)=0 \tag{3.2}$$

are obtained. Using the boundary conditions (2.21)-(2.22) the boundary conditions

$$X'(0)=0 \tag{3.3}$$

$$X'(1)+X(0)-X(1)=0 \tag{3.4}$$

are obtained. Since the function $u(x,t)$ is non-zero solution of the boundary value problem (2.20)-(2.23), the function $X(x)$ must be eigenfunction of the boundary value problem (3.2)-(3.4). Therefore it is required to examine the eigenvalue and eigenfunction problem for Sturm-Liouville problem (3.2)-(3.4). The basis problem of eigenfunctions of the boundary value problem (3.2)-(3.4) is important to verify the application of Fourier method to the boundary value problem (2.20)-(2.23). Because, it is come across the expansion problem to eigenfunctions of Sturm-Liouville problem corresponding to the initial function $\varphi(x)$ given in the condition (2.23) during the application of this method. Here it is used Theorem 1. Due to this theorem, regularity of the boundary conditions of the boundary value problem (3.2)-(3.4) and whether selfadjoint or not must be specified.

It is obtained the following relation for the numbers θ_{-1}, θ_0, θ_1 in the definition of regular boundary value conditions :

$$\frac{\theta_{-1}}{s}+\theta_0+\theta_1 s=\begin{vmatrix} i & -i \\ si & -\frac{i}{s}\end{vmatrix}=\frac{1}{s}-s$$

Here, it is clear that $\theta_{-1}=1,\ \theta_0=0,\ \theta_1=-1$ and since $\theta_{-1}\neq 0,\ \theta_1\neq 0$ and $\theta_0^2-4\theta_{-1}\theta_1\neq 0$, the boundary conditions (3.3), (3.4) are regular and strong regular (see [15], p.61).. Hence the boundary conditions in the boundary value problem (3.2)-(3.4) are regular and also strong regular.

Using Lagrange formula (see [15], p.9) let us construct boundary vale problem adjoint to the boundary vale problem (3.2)-(3.4). After simple calculations, adjoint boundary value problem is obtained in the form

$$Y''(x)+\lambda Y(x)=0,\ \ 0<x<1 \tag{3.5}$$

$$Y'(1)-Y(1)=0 \tag{3.6}$$

$$Y'(0)-Y(1)=0 \tag{3.7}$$

As you can see, because of that the boundary conditions of adjoint boundary value problem (3.3),(3.4) are not equivalent to the boundary conditions of the boundary value problem (3.6),(3.7), the boundary value problem (3.2)-(3.4) is not selfadjoint. Hence we get the following result from Theorem 1.

Corollary 3.1. The root functions of the boundary value problem (3.2)-(3.4) constitute Riesz basis in the space $L_2(0,1)$.

Since the boundary conditions of (3.2)-(3.4) are strong regular, the number of associated eigenfunctions may be finite. It can be shown that the boundary value problem (3.2)-(3.4) hasn't associated eigenfunctions and eigenvalues of this boundary value problem has the following form

$$\lambda_n' = \left[(2n-1)\pi\right]^2\left[1-\frac{4}{\left[(2n-1)\pi\right]^2}+O\left(\frac{1}{n^3}\right)\right], n=1,2,...,$$

$$\lambda_n'' = (2n\pi)^2,\ \ n=0,1,2,...,$$

and correspondig eigenfunctions respectively

$$X_{2n-1}(x)=\cos\left[(2n-1)\pi x\right]+O\left(\frac{1}{n}\right),\ \ n=1,2,...$$

$$X_{2n}(x)=\cos 2n\pi x,\ n=0,1,2,....$$

4. The Existence of Solution

Let's show the existence of classical solution of boundary value problems (2.8)-(2.11). Firstly, we consider the boundary value problem (2.20)-(2.23).

Let the sequence $\{X_n(x)\}_{n=0}^{\infty}$ be sequence of eigenfunctions corresponding eigenvalues $\{\lambda_n\}$ of this boundary value problem.

When $\lambda = \lambda_n$, it is obtained from the equation (3.1)

$$T_n(t) = \varphi_n e^{-\lambda_n t},$$

where, φ_n is arbitrary constant, $n = 0,1,2,\ldots$. Hence, special solutions of the equation (2.20) are found in the form

$$u_n(x,t) = X_n(x)T_n(t) = \varphi_n e^{-\lambda_n t} X_n(x).$$

Let us construct the following series

$$u(x,t) = \sum_{n=0}^{\infty} \varphi_n e^{-\lambda_n t} X_n(x). \tag{4.1}$$

By using initial condition (2.23), it is obtained

$$u(x,0) = \varphi(x) = \sum_{n=0}^{\infty} \varphi_n X_n(x). \tag{4.2}$$

It follows from Theorem 1 that the sequence $\{X_n(x)\}_{n=0}^{\infty}$ constitute Riesz basis in the space $L_2(0,1)$. Let $\{Y_n(x)\}_{n=0}^{\infty}$ eigenfunctions of adjoint boundary value problem be biorthogonal adjoint sequence. Then it is obtained from (4.2) that

$$\varphi_n(x) = (\varphi(x), Y_n(x))$$

and the series (4.1) is convergent.

Substituting special solutions $u_n(x,t)$ in the equation (2.20) and in the boundary conditions (2.21), (2.22), it follows that the equation and boundary conditions are satisfied. Shortly before the initial condition (2.23) has been satisfied.

Let us show that the series

$$\sum_{n=0}^{\infty}\varphi_n e^{-\lambda_n t}X_n(x),\ \sum_{n=0}^{\infty}\frac{\partial u_n}{\partial t},\ \sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2}$$

are uniform convergent for $t \geq \varepsilon > 0$. Taking into consideration asimptotik formulas for eigenvalues λ_n in [15] (see [15],say.64), these series are bounded from above by absolute convergent series $\sum_{n=0}^{\infty} M_n e^{-M_\varepsilon n^2}$, here $M_n > 0$ and $M_\varepsilon > 0$. Therefore according to Weierstrass criteria these series define continuous functions $u(x,t)$, $u_t(x,t)$, $u_{xx}(x,t)$ for $t \geq \varepsilon$. Let's show that the series (4.1) is uniform convergent on the domain $\overline{D}$. In fact, since $|X_n(x)| \leq c$ (here $0 < c < \infty$), then this series is bounded from above by numerical series $\sum_{n=1}^{\infty}|\varphi_n(x)|$. Using Cauchy-Schwarz inequality, we get

$$\sum_{n=0}^{\infty}|\varphi_n(x)| = \sum_{n=0}^{\infty}|\lambda_n||\varphi_n(x)|\frac{1}{|\lambda_n|} \leq \left(\sum_{n=0}^{\infty}|\lambda_n\varphi_n(x)|^2\sum_{n=0}^{\infty}\frac{1}{|\lambda_n|^2}\right)^{\frac{1}{2}} \quad (4.3)$$

Let us show that the series $\sum_{n=0}^{\infty}|\lambda_n\varphi_n(x)|^2$ is convergent. Since $\{Y_n(x)\}_{n=0}^{\infty}$ constitutes Riesz basis

$$\sum_{n=0}^{\infty}|\lambda_n\varphi_n(x)|^2 = \sum_{n=0}^{\infty}|\lambda_n(\varphi(x),Y_n(x))|^2 = \sum_{n=0}^{\infty}\left|-(\varphi(x),Y_n''(x))\right|^2$$
$$\leq \sum_{n=0}^{\infty}|(\varphi''(x),Y_n(x))|^2 \leq c\|\varphi''(x)\|^2 < \infty,$$

holds, here $c > 0$ and $|Y_n(x)| \leq c$. Therefore, the series (4.3) which bounds the series (4.1) from above is absolute convergent so the series (4.1) is uniform convergent on the domain $\overline{D}$ and defines continuous function $u(x,t)$.Hence, the following theorem is proved.

Theorem 4.1 : Assume that $\varphi(x) \in C^2[0,1]$ and the boundary conditions (3.3), (3.4) are satisfied. Then the function

$$u_1(x,t)=\sum_{n=0}^{\infty}\varphi_n(x)X_n(x)e^{-\lambda_n t}$$

İs classical solution of the boundary value problem (2.20)-(2.23), where the sequences $\varphi_n(x)=(\varphi(x),Y_n(x))$ and $\{Y_n(x)\}_{n=0}^{\infty}$ are biorthogonal adjoint sequences of sequence $\{X_n(x)\}_{n=0}^{\infty}$.

By knowing the solution of the boundary value problem (2.20)-(2.23), it is possible to get solution of the boundary value problem (2.16)-(2.19). This solution is obtained in the following form

$$u_2(x,t)=\sum_{n=0}^{\infty}\int_0^t f_n(\tau)e^{-\lambda_n(t-\tau)}d\tau X_n(x)$$

where

$$f_n(\tau)=\int_0^1 f(x,\tau)\overline{Y_n}(x)dx.$$

5. Conclusion

Non-classical boundary value problem has a classical solution on the domain D and this solution is found in the form $\upsilon(x,t)=u(x,t)+V(x,t)$, where $u(x,t)=u_1(x,t)+u_2(x,t)$ and $V(x,t)$ is expressed by (2.6).

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