

RESIDUE METHOD FOR THE SOLUTION OF HEAT EQUATION WITH NONLOCAL BOUNDARY CONDITION

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ABSTRACT

In this paper by using the residue method, the exact solution of mixed problem for linear heat equation with nonlocal boundary conditions is obtained. For this aim, at first, the formula for expansion of an arbitrary function in a series of residues of the solution of the corresponding spectral problem is proved. Further, this formula is used to show that if the mixed boundary-value problem in question has a sufficiently smooth solution, this solution can be represented by the given residue formula.

Key Words: *Residue method, Nonlocal boundary condition, Expansion formula, Residue representation*

LOKAL OLMAYAN SINIR KOŞULUNA SAHİP ISI DENKLEMİNİN REZİDÜ YÖNTEMİYLE ÇÖZÜMÜ

ÖZET

Makalede bir boyutlu ısı denklemi için yazılmış lokal olmayan sınır koşullu sınır-değer probleminin gerçek çözümü rezidü metodu ile elde edilmiştir. Bu amaçla herhangi bir sınırlı fonksiyonun, esas probleme karşılık gelen spektral problemin çözümünün tüm resudileri üzerinden ayrış formülü ispatlanmıştır. Ayrış formülü kullanılarak, esas problemin yeteri kadar pürüzsüz çözümünün varlığı çerçevesinde, bu çözümün önerilen rezidü formülü ile ifade edilebileceği gösterilmiştir.

1. INTRODUCTION

Many problems of engineering related to particle diffusion in turbulence plasma and also in heat distribution in a thin rod are reduced to the solution of the one dimension heat equation with nonlocal boundary conditions [1,2]. In reference [3] the problem of heat propagation in a thin rod in which one of the

boundary condition is given as $\int_0^{\ell} u(x, t) dx = \mu(t)$ is investigated. In the same work it is proved that the integral type condition is equivalent to the condition $\frac{\partial u(\ell, t)}{\partial x} - \frac{\partial u(0, t)}{\partial x} = 0$.

It is known that the operator generated by a heat equation with the nonlocal boundary conditions is not self adjoint. In this case the eigenfunctions of the corresponding spectral problem is not orthonormal and the question of expansion of any function in the series over these eigenfunctions is open. The various problems related with the initial boundary value problems of a heat equation with nonlocal conditions are studied in [4,5].

Within the limits of this paper, we will consider the following problem

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad (1)$$

$$u(x, 0) = \varphi(x), \quad (2)$$

$$\ell_1(u) \equiv \int_0^{\ell} q(x, t) dx = \mu(t), \quad (3)$$

$$\ell_2(u) \equiv \frac{\partial u(\ell, t)}{\partial x} = \nu(t), \quad (4)$$

where $q(x, t) = a \frac{\partial u(x, t)}{\partial x}$ is the flux function and $f(x, t)$, $\varphi(x)$, $\mu(t)$, $\nu(t)$ are known continuously differentiable functions on $[0, \ell]$, and $a > 0$ is given constant.

In order to obtain the classical solution of the problem (1)-(4), we reduce it to a problem with homogeneous boundary conditions corresponding to (3),(4). For

this purpose, we introduce a function $w(x, t) = \frac{\mu(t)}{a\ell}x$ which satisfies the boundary conditions (3) and (4), if the $\mu(t) = a\ell v(t)$ condition hold. Let us introduce the following substitution

$$u(x, t) = v(x, t) + w(x, t). \quad (5)$$

In this notation instead of (1)-(4), we have the following problem

$$\frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} + F(x, t), \quad (6)$$

$$v(x, 0) = g(x), \quad (7)$$

$$\ell_1(v) \equiv \int_0^\ell a \frac{\partial v(x, t)}{\partial x} dx = 0, \quad (8)$$

$$\ell_2(v) \equiv \frac{\partial v(\ell, t)}{\partial x} = 0 \quad (9)$$

The condition (8) can be written as

$$\ell_1(v) \equiv v(\ell, t) - v(0, t) = 0, \quad (10)$$

and $F(x, t) = f(x, t) - \frac{1}{a\ell} \frac{d\mu(t)}{dt}$, $g(x) = \varphi(x) - \frac{1}{a\ell} \mu(0)$. For the sake of simplicity, let $\ell = 1$, and $a = 1$.

It is known that the Fourier's method is a familiar means to find the exact solution of the mixed problem which is self adjoint and separable with respect to variable x and t . But, in this situation, it is necessary to expand any function $f(x) \in [0, 1]$ to the series over the eigenfunctions of the corresponding spectral problem, if the eigenfunctions are complete and orthogonal in $[0, 1]$.

If the problem is non-self adjoint, then the problem of expanding any function to series over these eigenfunctions is open. At first, A.L. Cauchy developed the residue method for the solution of boundary-value problem for the ordinary differential equations which are non-self adjoint.

The following development in the matter of residue method was investigated by M.L. Rasulov, in [7]. The outline of the residue method for solving mixed

problem is as follows. According to a definite procedure, a correspondence is made between a given mixed problem and some boundary-value problems with a complex parameter. With the aid of the solution of this spectral problem, we define a meromorphic function of a complex parameter whose complete residue gives a residue representation of formal solution of the given problem. Then, the formula for the expansion of an arbitrary function in series of residues of the solution of the spectral problem is used to show that, if the mixed boundary-value problem in question has a sufficiently smooth solution, it can be represented by the given residue formula.

In this paper, we will solve the problem (6)-(9) using the residue method developed in [7]. According to this, we will split the problem (6)-(9) into two auxiliary problems:

(i) The boundary-value problem

$$y'(x, \lambda) + \lambda^2 y(x, \lambda) = h(x), \quad (11)$$

$$y(0, \lambda) = y(1, \lambda), \quad y'(1, \lambda) = 0; \quad (12)$$

(ii) The Cauchy problem

$$\frac{\partial T(x, t, \lambda)}{\partial t} + \lambda^2 T(x, t, \lambda) = F(x, t), \quad (13)$$

$$T(x, 0, \lambda) = g(x), \quad (14)$$

here the function $h(x)$ is any function from $[0, 1]$. The problem (11),(12) is called spectral problem corresponding to the problem (6)-(9).

1.1 SOLUTION OF THE SPECTRAL PROBLEM

The exact solution of the spectral problem (11),(12) can be constructed as,

$$y(x, \lambda, h) = \int_0^\ell G(x, \xi, \lambda) h(\xi) d\xi, \quad (15)$$

(see [6,7]). Here $G(x, \xi, \lambda)$ is called the Green's function of the problem (11),(12), and has the following representation $G(x, \xi, \lambda) = \frac{\Delta(x, \xi, \lambda)}{\Delta(\lambda)}$,

$$\Delta(x, \xi, \lambda) = \begin{vmatrix} g(x, \xi, \lambda) & \sin \lambda x & \cos \lambda x \\ l_1(g(x, \xi, \lambda)) & l_1(\sin \lambda x) & l_1(\cos \lambda x) \\ l_2(g(x, \xi, \lambda)) & l_2(\sin \lambda x) & l_2(\cos \lambda x) \end{vmatrix}, \quad (16)$$

$$\Delta(\lambda) = \begin{vmatrix} l_1(\sin \lambda x) & l_1(\cos \lambda x) \\ l_2(\sin \lambda x) & l_2(\cos \lambda x) \end{vmatrix}, \quad (17)$$

$$g(x, \xi, \lambda) = \begin{cases} \frac{1}{\lambda} \sin \lambda(x - \xi), & 0 \leq \xi \leq x \leq 1 \\ -\frac{1}{\lambda} \sin \lambda(x - \xi), & 0 \leq x \leq \xi \leq 1, \end{cases} \quad (18)$$

$$\ell_1(\sin \lambda x) = \sin \lambda, \quad \ell_1(\cos \lambda x) = \cos \lambda - 1,$$

$$\ell_2(\sin \lambda x) = \lambda \cos \lambda, \quad \ell_2(\cos \lambda x) = -\lambda \sin \lambda,$$

$$\ell_1[g(x, \xi, \lambda)] = g(0, \xi, \lambda) - g(1, \xi, \lambda) = -\frac{1}{\lambda} [\sin \lambda(1 - \xi) - \sin \lambda \xi],$$

$$\ell_2[g(x, \xi, \lambda)] = \cos \lambda(1 - \xi),$$

and $\sin \lambda x$ and $\cos \lambda x$ are the fundamental functions of the problem (11),(12).

It is obvious that the eigenvalues of the problem (11),(12) are $\lambda_\nu = 2\pi\nu$, ($\nu = 0, \pm 1, \pm 2, \dots$) and corresponding eigenfunctions are $y_\nu(x) = \cos 2\pi\nu x$. It is known that the set of functions $\cos 2\pi\nu x$ is not complete and the problem to expand any function to series over these eigenfunctions is open.

In order to find the solution of the problem (11),(12) it is necessary to complement the set of the function $\cos 2\pi\nu x$ by adjoined functions, but it should be noted that this task is not easy.

Using the theorem of expansion of the any function on the series of residues of the solution of the spectral problem (11),(12), we claim that, if $f(x)$ is continuously differentiable in $[0, 1]$, then for any $x \in (0, 1)$ the following expansion formula

$$f(x) = \frac{-1}{2\pi\sqrt{-1}} \sum_{\nu} \int_{c_{\nu}} \lambda y(x, \lambda, f) d\lambda = - \sum_{\nu} \text{Res}_{\lambda_{\nu}} \int_0^{\ell} \lambda G(x, \xi, \lambda) f(\xi) d\xi \quad (19)$$

is valid, where c_{ν} is a closed contour enclosing only one pole of the function $\Delta(\lambda)$, (see [7]). Here, $\text{Res}_{\lambda_{\nu}} F(\lambda)$ denotes the integral residue of $F(\lambda)$ with respect to pole λ_{ν} , and the sum extend on all poles of $F(\lambda)$.

In order to find the explicit representation for the function $f(x)$, we will calculate the residue in (19) using the formula given in [8].

Let g and h be analytic at z_0 and let $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) = 0$, and $h''(z_0) \neq 0$, then $\frac{g(z)}{h(z)}$ has a second order pole at z_0 and the residue is

$$\text{Res}\left(\frac{g}{h}, z_0\right) = 2 \frac{g'(z_0)}{h''(z_0)} - \frac{2}{3} \frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}.$$

Now, we will calculate the residue of the function $\lambda e^{-\lambda^2 t} \frac{\Delta(x, \xi, \lambda)}{\Delta(\lambda)}$. In this case, $\Delta(\lambda_{\nu}) = 0$, $\Delta'(\lambda_{\nu}) = 0$, $\Delta''(\lambda_{\nu}) = 2\pi\nu \neq 0$, and $\Delta'''(\lambda_{\nu}) = 3$, and taking this above formula into account, we have

$$f(x) = 8 \sum_{\nu=1}^{\infty} \left\{ (1-x) \sin \lambda_{\nu} x \int_0^{\ell} f(\xi) \sin \lambda_{\nu} \xi d\xi + \cos \lambda_{\nu} x \int_0^{\ell} \xi f(\xi) \cos \lambda_{\nu} \xi d\xi \right\} + 2 \int_0^{\ell} \xi f(\xi) d\xi.$$

In order to construct the solution of the mixed problem (6)-(9), besides of the spectral problem (11),(12), we must use the solution of the problem (13),(14), which is

$$T(x, t, \lambda) = g(x) e^{-\lambda^2 t} + \int_0^t e^{-\lambda^2(t-\tau)} F(x, \tau) d\tau. \quad (20)$$

According to [7], we introduce the following operator, defined for any differentiable function $f(x)$ over $[0, 1]$ as

$$A_{\nu} f \equiv f_{\nu}^{(j)}(x) = \frac{-1}{2\pi\sqrt{-1}} \int_{c_{\nu}} \lambda^{2j+1} y(x, \lambda, f) d\lambda. \quad (21)$$

It is obvious that the formula (19) in notation (21) can be written as

$$-\sum_v f_v^{(0)}(x) = f(x). \quad (22)$$

When we apply the operator (21) to the equation (6), we get

$$\begin{aligned} \frac{-1}{2\pi\sqrt{-1}} \int_{c_v} \lambda^{2j+1} y\left(x, \lambda, \frac{\partial v}{\partial t}\right) d\lambda &= \frac{-1}{2\pi\sqrt{-1}} \int_{c_v} \lambda^{2j+1} y\left(x, \lambda, \frac{\partial^2 v}{\partial x^2}\right) d\lambda + \\ &\frac{-1}{2\pi\sqrt{-1}} \int_{c_v} \lambda^{2j+1} y(x, \lambda, F(x, t)) d\lambda. \end{aligned} \quad (23)$$

Since the operator defined by (21) is linear with respect to the third argument, we have

$$\frac{\partial v_v^{(j)}}{\partial t} = v_v^{(j+1)} + F_v^j, \quad (j = 0, 1, 2, \dots) \quad (24)$$

The initial condition for the equation (24) is

$$v_v^{(j)}(x, 0) = g_v^{(j)}(x). \quad (25)$$

It is not difficult to prove that the set functions

$$\tilde{v}_v^{(j)}(x, t) = \frac{-1}{2\pi\sqrt{-1}} \int_{c_v} \lambda^{2j+1} y(x, \lambda, T(x, t, \lambda)) d\lambda, \quad (26)$$

are solutions of the problem (24),(25). Here $T(x, t, \lambda)$ is the solution of the problem (13),(14). Indeed,

$$\begin{aligned} \frac{\partial \tilde{v}_v^{(j)}}{\partial t} &= \frac{-1}{2\pi\sqrt{-1}} \int_{c_v} \lambda^{2j+1} y\left(x, \lambda, \frac{\partial T}{\partial t}\right) d\lambda = \frac{-1}{2\pi\sqrt{-1}} \int_{c_v} \lambda^{2j+1} y(x, \lambda, -\lambda^2 T(x, \lambda, t)) d\lambda +, \\ &\frac{-1}{2\pi\sqrt{-1}} \int_{c_v} \lambda^{2j+1} y(x, \lambda, F(x, t)) d\lambda = \tilde{v}_v^{(j+1)} + F_v^j \end{aligned}$$

And

$$\tilde{v}_v^{(j)}(x, 0) = \frac{-1}{2\pi\sqrt{-1}} \int_{c_v} \lambda^{2j+1} y(x, \lambda, g(x)) d\lambda = g_v^{(j)}(x).$$

By virtue of uniqueness of the solution of the problem (6),(9) we have $v_v^{(j)}(x, t) = \tilde{v}_v^{(j)}(x, t)$. Since the solution $v(x, t)$ is continuously differentiable on $[0, 1]$, according to formula (22), we have

$$v(x, t) = \sum_v v_v^{(0)}(x, t) = -\frac{1}{2\pi\sqrt{-1}} \sum_v \int_{c_v} \lambda d\lambda \int_0^\ell G(x, \xi, \lambda) T(t, \lambda, \xi) d\xi. \quad (27)$$

Now, we must calculate the coefficients of the series (27). As it is mentioned

above that, the numbers $2\pi\nu$ ($\nu = \pm 1, \pm 2, \dots$) are poles of the Green's function, besides this numbers are the poles of second order of the function $G(x, \xi, \lambda)$, that is $\Delta(\lambda_\nu) = 0$, $\Delta'(\lambda_\nu) = 0$ and $\Delta''(\lambda_\nu) \neq 0$. Moreover, $\lambda_0 = 0$ is the pole of third order of the Green's function.

Calculating the residue of the function $\lambda e^{-\lambda^2 t} \frac{\Delta(x, \xi, \lambda)}{\Delta(\lambda)}$, for the solution

$v(x, t)$, we have the following representation

$$\begin{aligned}
 v(x, t) = & 32\pi \sum_{\nu=1}^{\infty} \nu e^{-(2\pi\nu)^2 t} \cos(2\pi\nu x) \int_0^{\ell} \sin(2\pi\nu\xi) g(\xi) d\xi \\
 & - 8 \sum_{\nu=1}^{\infty} e^{-(2\pi\nu)^2 t} \left\{ (1-x) \sin(2\pi\nu x) \int_0^{\ell} \sin(2\pi\nu\xi) g(\xi) d\xi \right\} \\
 & - 8 \sum_{\nu=1}^{\infty} e^{-(2\pi\nu)^2 t} \left\{ \cos(2\pi\nu x) \int_0^{\ell} \xi \cos(2\pi\nu\xi) g(\xi) d\xi \right\} \\
 & + 32\pi \sum_{\nu=1}^{\infty} \nu \cos(2\pi\nu x) \int_0^{\ell} \sin(2\pi\nu\xi) \left\{ \int_0^t (t-\tau) e^{-(2\pi\nu)^2 (t-\tau)} f(\xi, \tau) d\tau \right\} d\xi \\
 & - 8 \sum_{\nu=1}^{\infty} \int_0^t e^{-(2\pi\nu)^2 (t-\tau)} \left\{ (1-x) \sin(2\pi\nu x) \int_0^{\ell} \sin(2\pi\nu\xi) d\xi \right\} f(\xi, \tau) d\tau \\
 & - 8 \sum_{\nu=1}^{\infty} \int_0^t e^{-(2\pi\nu)^2 (t-\tau)} \left\{ \cos(2\pi\nu x) \int_0^{\ell} \xi \cos(2\pi\nu\xi) d\xi \right\} f(\xi, \tau) d\tau \\
 & - 4 \int_0^{\ell} \xi \left\{ \tau g(\tau) - \int_0^t \tau f(\xi, \tau) d\tau \right\} d\xi. \tag{28}
 \end{aligned}$$

The obtained solution in fact provides us to prove the theorem of uniqueness and existence. On the other hand, using this solution, we can estimate the numerical solutions of the investigated problem. Furthermore, the formula (28) composes the basis of approximate solution of the problem (1)-(3) too.

2. APPLICATION RESIDUE METHOD TO THE DRICHLET'S PROBLEM FOR A HEAT EQUATION

In this section we will investigate the first initial boundary problem for the heat equation in $[0,1]$ as

$$v(x,t) = \sum_v v_v^{(0)}(x,t) = \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (29)$$

$$u(x,0) = \varphi(x), \quad (30)$$

$$\ell_1(u) \equiv u(0,t) = 0, \quad \ell_2(u) \equiv u(1,t) = 0. \quad (31)$$

In order to find the solution of the problem (29)-(31) at first we will use the Fourier's method, then we will apply the residue method to the same problem.

It is known that [9] the solution of the problem (29)-(31) obtained by Fourier's method in the following form

$$u(x,t) = \sum_{v=1}^{\infty} \varphi_v e^{-(\pi v)^2 t} \sin \pi v x, \quad (32)$$

where the Fourier's coefficients φ_v are

$$\varphi_v = \frac{2}{\ell} \int_0^{\ell} \varphi(\xi) \sin \pi v \xi d\xi. \quad (33)$$

Now, we will find the solution of the problem (29)-(31) using the residue method. In accordance with the general schema of the residue method, the following spectral problem

$$y'' + \lambda^2 y = h(x), \quad (34)$$

$$y(0) = 0, \quad y(1) = 0 \quad (35)$$

is introduced. The solution of the problem (34),(35) is in the form as (8). In this case, the Green's function $G_1(x, \xi, \lambda)$ is in the form

$$G_1(x, \xi, \lambda) = \frac{\Delta_1(x, \xi, \lambda)}{\Delta_1(\lambda)}, \text{ where}$$

$$\Delta_1(x, \xi, \lambda) = \begin{vmatrix} g(x, \xi, \lambda) & \sin \lambda x & \cos \lambda x \\ l_1(g(x, \xi, \lambda)) & l_1(\sin \lambda x) & l_1(\cos \lambda x) \\ l_2(g(x, \xi, \lambda)) & l_2(\sin \lambda x) & l_2(\cos \lambda x) \end{vmatrix}, \quad (36)$$

$$\Delta_1(\lambda) = \begin{vmatrix} l_1(\sin \lambda x) & l_1(\cos \lambda x) \\ l_2(\sin \lambda x) & l_2(\cos \lambda x) \end{vmatrix}, \quad (37)$$

$$g(x, \xi, \lambda) = \begin{cases} \frac{1}{\lambda} \sin \lambda(x - \xi), & 0 \leq \xi \leq x \leq 1 \\ -\frac{1}{\lambda} \sin \lambda(x - \xi), & 0 \leq x \leq \xi \leq 1, \end{cases} \quad (38)$$

and

$$\begin{aligned} \ell_1(\sin \lambda x) &= 0, \quad \ell_1(\cos \lambda x) = 1, \\ \ell_2(\sin \lambda x) &= \sin \lambda, \quad \ell_2(\cos \lambda x) = \cos \lambda, \\ \ell_1[g(x, \xi, \lambda)] &= \frac{\sin \lambda \xi}{\lambda}, \\ \ell_2[g(x, \xi, \lambda)] &= \frac{\sin \lambda(1 - \xi)}{\lambda}, \end{aligned}$$

and $h(x)$ is an arbitrary function from $L_2[0, 1]$.

2.1 FORMULA OF EXPANSION

It is obvious that the function (8) is the solution of the problem (34),(35) for any points except the roots of the $\Delta_1(\lambda) = 0$, which are the eigenvalues of the spectral problem (34),(35). The roots of this equation are $\lambda_\nu = \pi\nu$, ($\nu = 0, \pm 1, \pm 2, \dots$).

Let ℓ_k ($k=1, 2$) be the bounded operators, which define the boundary conditions (35). The function

$$y(x, \lambda) = \sum_{i=1}^2 c_i y_i(x, \lambda)$$

satisfies the following conditions

$$\ell_k(y(x, \lambda)) = 0, \quad (k = 1, 2). \quad (39)$$

Here $y_i(x, \lambda)$ are the set of fundamental solutions of the equation (34), that is

$$y_1 = \sin \lambda x, \quad y_2 = \cos \lambda x.$$

Since the equation (39) is the linear algebraic systems of equations with respect to c_1 and c_2 , then the condition

$$\Delta_1(\lambda) = \begin{vmatrix} 0 & 1 \\ \sin \lambda & \cos \lambda \end{vmatrix} = 0, \quad (40)$$

is necessary and sufficient for the unique solution of (39).

It is obvious that the function $\Delta_1(\lambda)$ has countable real roots λ_ν of which $\lambda_\nu \rightarrow \infty$, as $\nu \rightarrow \infty$.

It is easy to show that the set of the functions $y_\nu = \sin \pi \nu x$, ($\nu = 0, 1, 2, \dots$) are complete. In this situation, an arbitrary function $h(x) \in L_2[0, 1]$ can be expanded to the Fourier's series as,

$$h(x) = \sum_{\nu=1}^{\infty} (h, y_\nu) y_\nu, \quad (41)$$

and the sum of this series approaches to $h(x)$ in the sense of L_2 , here

$$(h, y_\nu) = \int_0^1 h(\xi) y_\nu(\xi) d\xi. \quad (42)$$

Let us introduce the notation

$$\mathfrak{S}(\lambda)(h(x)) = \int_0^1 G(x, \xi, \lambda) h(\xi) d\xi. \quad (43)$$

Since

$$\mathfrak{S}y_\nu = \lambda_\nu y_\nu = \bar{\ell} y_\nu + (\lambda_\nu - \bar{\ell}) y_\nu,$$

we can show that

$$(\mathfrak{S}(\ell)f(x), y_\nu) = (f, \mathfrak{S}(\bar{\ell})y_\nu) = (\lambda_\nu - \bar{\ell})^{-1} (f, y_\nu). \quad (44)$$

It is obvious that the Fourier's series of the function defined by (43) is

$$\sum_{\nu=0}^{\infty} (\Im(\ell) f, y_{\nu}) y_{\nu} = \sum_{\nu=0}^{\infty} (\lambda_{\nu} - \ell)^{-1} (f, y_{\nu}) y_{\nu}. \quad (45)$$

Using the residue formula, we have

$$\frac{-1}{2\pi\sqrt{-1}} \int_{c_{\nu}} \Im(\lambda) f d\lambda = \sum_{\nu=0}^{\infty} (f, y_{\nu}) y_{\nu}. \quad (46)$$

Here c_{ν} is a closed contour enclosing only one pole of $\Delta(\lambda)$.

Then, the solution of the problem (29)-(31) is

$$\begin{aligned} u(x, t) &= \sum_{\nu} \int_{c_{\nu}} \lambda y(x, \lambda, T(x, \lambda, t)) d\lambda = - \sum_{\nu} \int_0^{\ell} \frac{\lambda_{\nu} e^{-\lambda_{\nu}^2 t} \Delta_1(x, \xi, \lambda_{\nu})}{\Delta_1'(\lambda_{\nu})} \varphi(\xi) d\xi \\ &= 2 \sum_{\nu} e^{-(\pi \nu)^2 t} \sin \pi \nu x \int_0^{\ell} \varphi(\xi) \sin \pi \nu \xi d\xi. \end{aligned}$$

As a conclusion, we have shown that the both solutions of the problem (1)-(3) obtained by the Fourier's and residue methods coincide.

3. CONCLUSION

The residue method for solving a non-self adjoint problem in heat propagation

with the boundary condition $\int_0^{\ell} q(x, t) d\xi = \mu(t)$, where

$q(x, t) = a \frac{\partial u(x, t)}{\partial x}$, and $u(x, t)$ is unknown and $\mu(t)$ is a given function

has been investigated. The representation for the solution of this problem in the form of rapidly decreasing series which is efficient from a practical and computational point of views has been obtained. Further, the residue method has been applied to obtain the solution of first kind initial boundary value problem of the heat equation.

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