Journal of Science and Technology
2 (2), 2008, 206-217

ON THE B-SCROLLS WITH TIME-LIKE DIRECTRIX IN 3-DIMENSIONAL MINKOWSKI SPACE E_1^3

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Received: 25 January 2008, Accepted: 28 April 2008

ABSTRACT

In this paper, as special ruled surfaces, b-scrolls with time-like directrix are introduced in 3-dimensional Minkowski space E_1^3, [1] and [3]. The generating vector of b-scroll is space-like V^3 binormal vector of time-like directrix curve. The Normal vector, the matrix corresponding to the shape operator, the Gaussian and mean curvatures, I. and II. Fundamental forms, asymptotic lines and curvature lines of b-scrolls together with striction space are studied.

Key words *:B-scroll, time-like, ruled surfaces, shape operator.*

ÖZET

E_1^3 $3-$boyutlu Mikowski uzayında $\eta(I)$ time-like dayanak eğrisi boyunca, bu eğrinin space-like V^3 binormal vektörü tarafından üretilen bir regle yüzey olarak b-scroll tanımlandı [1] ve [3]. Bu yüzeyin S şekil operatörüne karşılık gelen matris, Gauss eğriliği, ortalama eğriliği hesaplandı. I. ve II. temel formlar, asimptotik çizgileri and eğrilik çizgilerini veren denklemler ifade edildi.

1. INTRODUCTION

Let $\eta(I)$ be a time-like curve with arc length t in 3-dimensional Minkowski space $\mathbf{E}_1^3$.If $\dot{\eta}(t) = V_1$ is a time-like vector then $\eta(I)$ is a time-like curve. That is $\langle V_1, V_1 \rangle = -1$, [7]. The Frenet vectors of $\eta(I)$ are V_1, V_2, V_3, where V_2 normal vector is space-like and binormal vector is space-like V_3. Then

$$\begin{aligned} \langle V_2, V_2 \rangle &= \langle V_3, V_3 \rangle = 1 \\ \langle V_1, V_3 \rangle &= \langle V_2, V_3 \rangle = \langle V_1, V_2 \rangle = 0 \end{aligned}$$

hold.

In 3-dimensional Minkowski space $\mathbf{E}_1^3$, natural lorentz metric is

$$\langle \ldots, \ldots \rangle = (-,+,+)$$

so the cross product of

$$\vec{a} = (a_1, a_2, a_3) \text{ and } \vec{b} = (b_1, b_2, b_3) \text{ is}$$

$$\vec{a} \wedge \vec{b} = (a_2 b_3 - a_3 b_2, a_1 b_3 - a_3 b_1, a_2 b_1 - a_1 b_2),$$

or this product is given by

$$\vec{a} \wedge \vec{b} = -\det \begin{bmatrix} -V_1 & V_2 & V_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

or in opposite direction

$$\vec{a} \wedge \vec{b} = \det \begin{bmatrix} -V_1 & V_2 & V_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \tag{1}$$

can be used, [9].

Since V 1 is time-like vector, V 2 and V 3 are space-like vectors in 3-dimensional Minkowski space E_1^3, Frenet Formulas, ([2], [4] and [5]), can be given by the following equations ,

$$\begin{aligned} \dot{V}_1 &= k_1 V_2 \\ \dot{V}_2 &= k_1 V_1 + k_2 V_3 \\ \dot{V}_3 &= -k_2 V_2 \end{aligned}$$

is the matrix form of which is
$$\begin{bmatrix}\dot{V}_1\\ \dot{V}_2\\ \dot{V}_3\end{bmatrix}=\begin{bmatrix}0 & k_1 & 0\\ k_1 & 0 & k_2\\ 0 & -k_2 & 0\end{bmatrix}\begin{bmatrix}V_1\\ V_2\\ V_3\end{bmatrix}. \tag{2}$$

2. THE B-SCROLLS WITH TIME-LIKE DIRECTRIX IN 3-DIMENSIONAL MINKOWSKI SPACE E_1^3

Let $\eta(I)$ be a time-like curve with arc length t, and let V_1, V_2, V_3 be the Frenet vectors. The parametrization of b-scroll whose directrix is the time-like curve $\eta(I)$ and generating vector is the space-like binormal vector V ^{3,} is

$$\varphi(t,u)=\eta(t)+uV_3(t) \tag{3}$$

in 3-dimensional Minkowski space $\mathbf{E}_1^3$. Let M be the surface whose ordered basis vectors φ_t and φ_u at the point $\eta(t)$ are given by

$$\begin{aligned}\varphi_t &= V_1-uk_2V_2\\ \varphi_u &= V_3\end{aligned}$$

Denote the asymptotic bundel by $A(t)=Sp\{V_2,V_3\}$ is and the tangentian bundel by $T(t)=Sp\{V_1,V_2,V_3\}$. Because of

$\dim A(t)\neq \dim T(t),$

there is not an edge space but there is a striction (curve) space, [8]. Let p(t) is any curve with equation

$$p(t)=\eta(t)+u(t)V_3(t) \tag{4}$$

on surface M. Differantiating $p(t)$ with respect to t

$$\begin{aligned}\dot{p}(t) &= \dot{\eta}+\dot{u}V_3+u\dot{V}_3\\ &= V_1+\dot{u}V_3-uk_2V_2\end{aligned}$$

is obtained. The solution vectors u of the equation

$$\left\langle \dot{p}(t),\frac{d}{dt}\left[u(t)V_3(t)\right]\right\rangle=\left\langle V_1+\dot{u}V_3-uk_2V_2,\dot{u}V_3-uk_2V_2\right\rangle=0 \tag{5}$$

are the position vectors of striction (curve) space[10].This equation implies $\dot{u}^2=0$ and $u^2k_2^2=0$. If $\dot{u}=0,$ then u is constant. Hence, if $u=0$,then striction curve is $\eta(I)$ itself. If $k_2=0$, then striction curve does not have a torsion. That is, striction curve is a plane curve.

The parametrization of surface M is

$$\varphi(t,u)=\eta(t)+uV_3(t)$$

Since

$$\overline{\varphi_t}=\frac{V_1-uk_2V_2}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}}\quad and\ \overline{\varphi_u}=\frac{\varphi_u}{\|\varphi_u\|}=\varphi_u=V_3$$

then

$$\left\langle\overline{\varphi_t},\overline{\varphi_u}\right\rangle=0.$$

That is $\overline{\varphi_t},\overline{\varphi_u}$ are ortonormal tangent vectors and $\chi(M)=Sp\left\{\overline{\varphi_t},\overline{\varphi_u}\right\}$. The Normal vector of the surface M is

$$N=\frac{\overline{\varphi_t}\wedge\overline{\varphi_u}}{\left\|\overline{\varphi_t}\wedge\overline{\varphi_u}\right\|} \tag{6}$$

$$N=\det\begin{bmatrix}-V_1 & V_2 & V_3\\ \dfrac{1}{(-1+u^2k_2^2)^{\frac{1}{2}}} & \dfrac{-uk_2}{(-1+u^2k_2^2)^{\frac{1}{2}}} & 0\\ 0 & 0 & 1\end{bmatrix}\quad;\ \left\|\overline{\varphi_t}\wedge\overline{\varphi_u}\right\|=1.$$

Therefore we obtain

$$N=\frac{uk_2V_1-V_2}{\left(-1+u^2k_2^2\right)^{\frac{1}{2}}}. \tag{7}$$

Using this Normal vector we can find the matrix S corresponding to the shape operator. We know that

$$S(\overline{\varphi_t})=\lambda_1\overline{\varphi_t}+\lambda_2\overline{\varphi_u}\quad\Rightarrow\lambda_1=\left\langle S(\overline{\varphi_t}),\overline{\varphi_t}\right\rangle$$

$$\Rightarrow\lambda_2=\left\langle S(\overline{\varphi_t}),\overline{\varphi_u}\right\rangle$$

$$S(\overline{\varphi_u})=\mu_1\overline{\varphi_t}+\mu_2\overline{\varphi_u}\quad\Rightarrow\mu_1=\left\langle S(\overline{\varphi_u}),\overline{\varphi_t}\right\rangle$$

$$\Rightarrow\mu_2=\left\langle S(\overline{\varphi_u}),\overline{\varphi_u}\right\rangle$$

Using the following formula

$$S(\overline{\varphi_t})=\frac{1}{\|\varphi_t\|}\frac{\partial N}{\partial t}=\frac{1}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}}\frac{\partial N}{\partial t}, \tag{8}$$

under the condition $u^2k_2^2-1>0$, we have

$$S(\overline{\varphi_t}) = \frac{\left(k_1 - u\dot{k}_2 - u^2 k_1 k_2^2\right)V_1 + \left(u^2 k_2 \dot{k}_2 - u k_1 k_2 + u^3 k_1 k_2^3\right)V_2 + \left(k_2 - u^2 k_2^3\right)V_3}{(u^2 k_2^2 - 1)^2}.$$

The values of λ_1 and λ_2 are found as

$$\lambda_1 = \left\langle S(\overline{\varphi_t}), \overline{\varphi_t} \right\rangle = \left\langle S(\overline{\varphi_t}), \frac{V_1 - u k_2 V_2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} \right\rangle = \frac{k_1 - u\dot{k}_2 - u^2 k_1 k_2^2}{\left(u^2 k_2^2 - 1\right)^{\frac{3}{2}}} \tag{9}$$

and

$$\lambda_2 = \left\langle S(\overline{\varphi_t}), \overline{\varphi_u} \right\rangle = \frac{-k_2}{u^2 k_2^2 - 1} \tag{10}$$

respectively. Using the formula

$$S(\overline{\varphi_u}) = S(\varphi_u) = \frac{\partial N}{\partial u} \tag{11}$$

we compute

$$\frac{\partial N}{\partial u} = -\frac{-k_2 V_1 \sqrt{u^2 k_2^2 - 1} + \left(u k_2 V_1 - V_2\right)\left(\sqrt{u^2 k_2^2 + 1}\right)'}{u^2 k_2^2 - 1}$$

$$S(\overline{\varphi_u}) = \frac{-k_2 V_1 + u k_2^2 V_2}{\left(u^2 k_2^2 - 1\right)^{\frac{3}{2}}}.$$

Thus, for

$$\mu_1 = \left\langle S(\overline{\varphi_u}), \overline{\varphi_t} \right\rangle \tag{12}$$

$$\mu_2 = \left\langle S(\overline{\varphi_u}), \overline{\varphi_u} \right\rangle \tag{13}$$

the value of μ_1 is

$$\mu_1 = \frac{-k_2}{\left(u^2 k_2^2 - 1\right)} = \lambda_2$$

and the value of μ_2 is

$$\mu_2 = 0.$$

As a result, the matrix corresponding to the shape operator S is

$$S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & 0 \end{bmatrix} \tag{14}$$

$$S = \begin{bmatrix} \dfrac{k_1 - u\dot{k}_2 - u^2 k_1 k_2^2}{\left(u^2 k_2^2 - 1\right)^{\frac{3}{2}}} & \dfrac{-k_2}{u^2 k_2^2 - 1} \\ \dfrac{-k_2}{u^2 k_2^2 - 1} & 0 \end{bmatrix} \tag{15}$$

3. GAUSSIAN CURVATURE:

Gaussian curvature of b-scroll whose directrix is time-like curve and generating vector is space-like binormal vector, is denoted by K ,whic is non negative [6], in 3-dimensional Minkowski space E_1^3

$$\begin{aligned} K &= \varepsilon_1 \det S = \det S \\ &= \frac{k_2^2}{\left(u^2 k_2^2 - 1\right)^2} \end{aligned} \tag{16}$$

where

$\varepsilon_1 = \langle N, N \rangle = -1$

and N is the time-like Normal of bscroll.

4. MEAN CURVATURE

Mean curvature of b-scroll whose directrix is time-like curve and generating vector is space-like binormal vector is denoted by H in 3-dimensional Minkowski space E_1^3

$$\begin{aligned} H &= \mathrm{tr} S \\ &= \lambda_1 = \frac{k_1 - u\dot{k}_2 - u^2 k_1 k_2^2}{\left(u^2 k_2^2 - 1\right)^{\frac{3}{2}}} \end{aligned} \tag{17}$$

5. I. FUNDAMENTAL FORM:

I.Fundamental form of b-scroll whose directrix is time-like curve and generating vector is space-like binormal vector, is defined by

$$I = \langle d\varphi, d\varphi \rangle \text{and } d\varphi = \varphi_t dt + \varphi_u du, \tag{18}$$

hence ,

$I = \langle \varphi_t, \varphi_t \rangle dtdt + 2\langle \varphi_t, \varphi_u \rangle dtdu + \langle \varphi_u, \varphi_u \rangle dudu$

which results in

$I = \left(u^2 k_2^2 - 1\right) dtdt + dudu$

,thus we can write this quadratic form as the matrix form

$$\bar{I} = \begin{bmatrix} u^2k_2^2 - 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{19}$$

$\det \bar{I} = u^2k_2^2 - 1.$

6. II. FUNDAMENTAL FORM:

II.Fundamental form of b-scroll whose directrix is time-like curve and generating vector is space-like binormal vector is

$$II = \langle S(d\varphi), d\varphi \rangle \text{ with } d\varphi = \varphi_t dt + \varphi_u du \tag{20}$$

from which

$$\begin{aligned} II &= \langle S(\varphi_t)dt + S(\varphi_u)du, \varphi_t dt + \varphi_u du \rangle \\ &= \langle S(\varphi_t), \varphi_t \rangle dtdt + \langle S(\varphi_t), \varphi_u \rangle dtdu \\ &\quad + \langle S(\varphi_u), \varphi_t \rangle dudt + \langle S(\varphi_u), \varphi_u \rangle dudu \end{aligned}$$

is computed and hence

$$II = \lambda_1 \left(u^2k_2^2 - 1\right)^{\frac{1}{2}} dtdt + 2\lambda_2 \left(u^2k_2^2 - 1\right)^{\frac{1}{2}} dtdu + 0dudu$$

resulting in

$$II = \frac{k_1 - u\dot{k}_2 - u^2k_1k_2^2}{\left(u^2k_2^2 - 1\right)^{\frac{1}{2}}} dtdt + \frac{-2k_2}{\left(u^2k_2^2 - 1\right)^{\frac{1}{2}}} dtdu.$$

We can write this quadratic form as a matrix

$$\overline{II} = \begin{bmatrix} \dfrac{k_1 - u\dot{k}_2 - u^2k_1k_2^2}{\left(u^2k_2^2 - 1\right)^{\frac{1}{2}}} & \dfrac{-k_2}{\left(u^2k_2^2 - 1\right)^{\frac{1}{2}}} \\ \dfrac{-k_2}{\left(u^2k_2^2 - 1\right)^{\frac{1}{2}}} & 0 \end{bmatrix} \tag{21}$$

and

$$\det \overline{II} = \frac{-k_2^2}{u^2k_2^2 - 1}.$$

7. ASYMPTOTIC LINES:

In 3-dimensional Minkowski space E_1^3, asymtotic lines of b-scroll whose directrix is time-like curve and generating vector is space-like binormal vector, are curves with the following equation

$$II = \langle S(d\varphi), d\varphi \rangle = 0 \tag{22}$$

$$II = \frac{k_1 - u\dot{k}_2 - u^2 k_1 k_2^2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} dtdt + \frac{-2k_2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} dtdu = 0.$$

It can be shown that this equation becomes

$$\left(\frac{k_1 - u\dot{k}_2 - u^2 k_1 k_2^2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} dt + \frac{-2k_2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} du \right) dt = 0 \tag{23}$$

after some computations. Therefore, we investigate the following two cases:
i) If $dt = 0$, then asymtotic lines are the mainlines with equation $t = c_1$.
ii) otherwise the asymtotic lines have the following equation

$$\frac{k_1 - u\dot{k}_2 - u^2 k_1 k_2^2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} dt + \frac{-2k_2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} du = 0. \tag{24}$$

8. CURVATURE LINES:

In 3-dimensional Minkowski space E_1^3, if T is a tanget vector of any curve in surface M we know that $T \in \chi(M) = Sp\{\varphi_t, \varphi_u\}$ and differantial equation of Curvature lines is $\overline{S}T = \lambda T$. First, we find tanget vector T .From which

$$\begin{aligned} d\varphi &= \varphi_t dt + \varphi_u du \\ &= \left(V_1 - uk_2 V_2\right) dt + V_3 du \end{aligned}$$

$$T = \frac{V_1 - uk_2 V_2}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} dt + V_3 du \tag{25}$$

is computed and hence

$$T = \frac{dt}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} \varphi_t + du\varphi_u$$

The matrix form of tanget vector T is

$$T = \begin{bmatrix} \dfrac{dt}{\left(u^2 k_2^2 - 1\right)^{\frac{1}{2}}} \\ du \end{bmatrix}.$$

Replacing this value on the equation $\overline{S}T = \lambda T$,we get

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{dt}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}} \\ du \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda dt}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}} \\ \lambda du \end{bmatrix}. \tag{26}$$

It can be shown that this equation becomes

$$\left.\begin{aligned} \frac{\lambda_1-\lambda}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}}dt+\lambda_2 du=0 \\ \frac{\lambda_2}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}}dt-\lambda du=0 \end{aligned}\right\} \tag{27}$$

after some computations.Therefore we investigate the following equations

$$\left.\begin{aligned} \lambda(\lambda_1-\lambda)\frac{dt}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}}+\lambda_2^2\frac{dt}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}}=0 \\ \left[\lambda(\lambda_1-\lambda)+\lambda_2^2\right]\frac{dt}{\left(u^2k_2^2-1\right)^{\frac{1}{2}}}=0 \end{aligned}\right\}$$

where there is two results

i) If $dt=0$, curvature lines are the mainlines with equation $\Rightarrow t=c_1$.

ii) Otherwise, the quadratic equation $-\lambda^2+\lambda_1\lambda+\lambda_2^2=0$, since $\Delta_\lambda=\lambda_1^2+4\lambda_2^2$ is always positive, we have two distinct roots

$$\lambda=\frac{-\lambda_1\mp\sqrt{\lambda_1^2+4\lambda_2^2}}{2\lambda_1} \tag{28}$$

If we replace λ_1 and λ_2 in the equation,then the other curvature lines have been obtained.

Example1: In 3-dimensional Minkowski space E_1^3

$$\eta(t)=(\sqrt{2}t,\cos t,\sin t)$$

is a time-like curve.The parametrization of b-scroll with directrix $\eta(t)$ and generating $V_3(t)$ is

$$\begin{aligned} \varphi(t,u) &= \eta(t)+uV_3(t) \\ \varphi(t,u) &= (\sqrt{2}t-u,\cos t+u\sqrt{2}\sin t,\sin t-u\sqrt{2}\cos t) \end{aligned}$$

Here $V_3(t)$ is space-like binormal vector of the time-like curve $\eta(t)$.

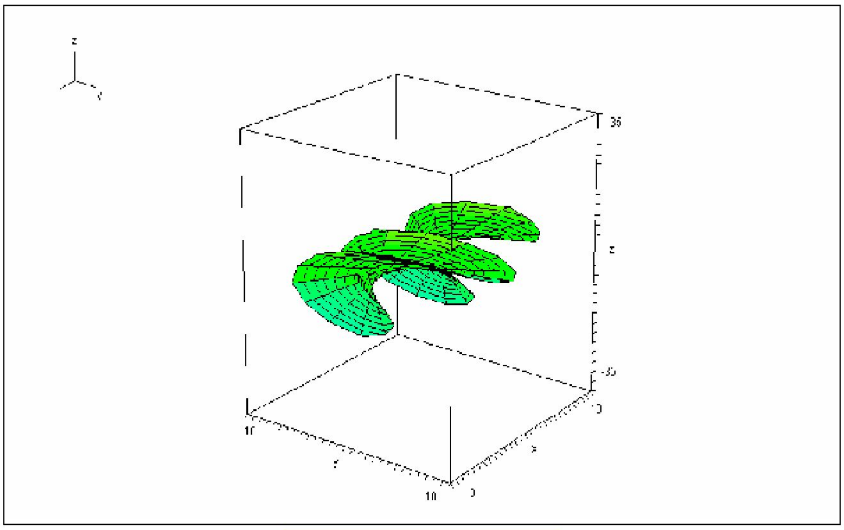

Figure 1 : Graph of 1st Example
[11]

Example 2: In 3-dimensional Minkowski space E_1^3

$$\eta(t) = (t + \frac{t^3}{6}, \frac{t^2}{2}, -\frac{t^3}{6})$$

is a time-like curve.The parametrization of b-scroll with directrix $\eta(t)$ and generating $V_3(t)$ is

$$\varphi(t,u) = \eta(t) + uV_3(t)$$

$$\varphi(t,u) = (t + \frac{t^3}{6} + u\frac{t^2}{2}, \frac{t^2}{2} + ut, -\frac{t^3}{6} + u - u\frac{t^2}{2})$$

Here $V_3(t)$ is space-like binormal vector of the time-like curve.

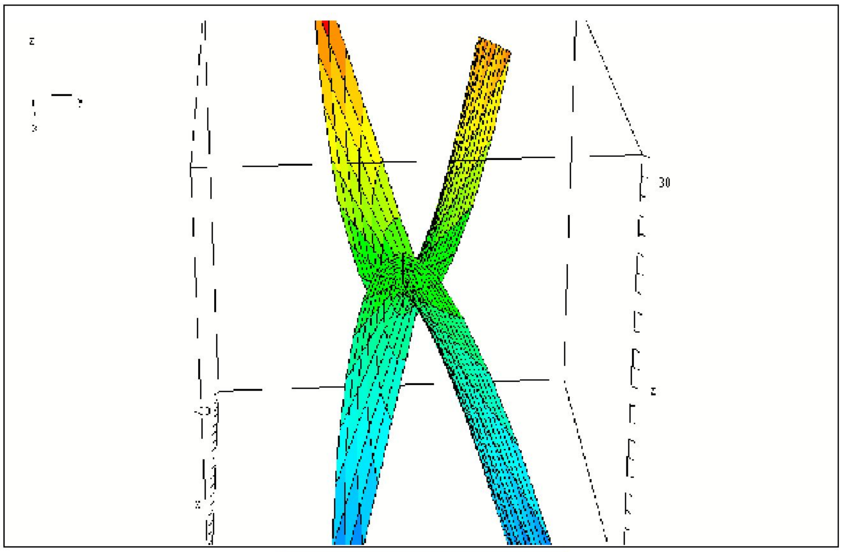

Figure2 : Graph of 2nd Example

[11]

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